Reconstruction of Irrotational Fields using Longitudinal Line Integrals: Application in Inverse EEG

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Outline

EEG Applications

Aim: Estimation of the brain activity by non-invasive measurements along the scalp surface.

- Diagnostic tool in neurology, e.g., epileptic seizures, brain death and neuroscience.
- Brain–Computer Interface (BCI) or Mind–Machine Interface (MMI),
- Estimate Cognitive States using EEG.

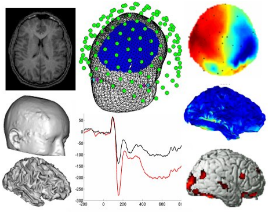
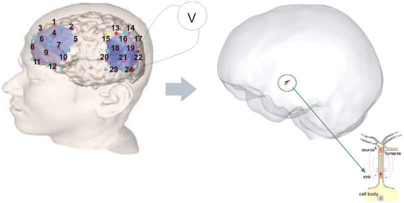


Image: Dr. R. Henson, Univ. of Cambridge

Inverse Bioelectric Field Problem

Reconstruction of the primary current sources distributions $\bar{J}^i(x, y, z, t)$ in the bounded domain (brain) using the EEG recording.



A primary current is generated by the electrochemical processes in the excited nerve cells (Conversion of energy from chemical to electric form). Assumption: The individual cells behave like electric current dipoles: $M_q(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_q)$.

Forward/Inverse EEG Modeling

Definition (Mathematical Formulation of Inverse EEG Problem)

Let $\sigma(\mathbf{r})$ be the conductivity and $\Phi|_{\partial\Omega}$ are given potential measurements on the surface of a bounded domain $\Omega \subset \Re^3$, estimate the current sources $\nabla \cdot \mathbf{J}^{prim}$ which satisfy the Poisson equation :

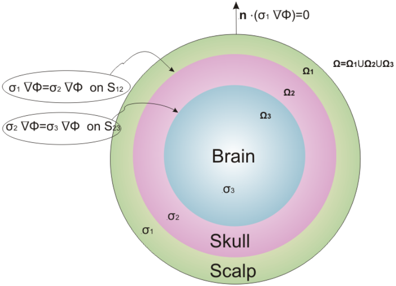
$$\nabla \cdot \sigma \nabla \Phi = \nabla \cdot \mathbf{J}^{prim}$$

with boundary conditions

$$\mathbf{n} \cdot \sigma \nabla \Phi = 0 \text{ on } \partial\Omega \text{ (no-outward flux)}$$

$$\int_{\partial\Omega} \Phi dS = 0 \text{ (fix ground potential)}$$

Basic Assumptions



Common head model: three compartments and isotropic conductivities σ.

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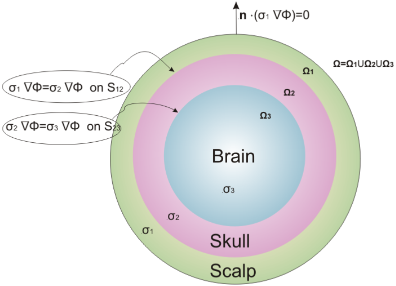

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  Capacitance component of the tissues and electromagnetic wave effects are neglected. Thus, at an instant,
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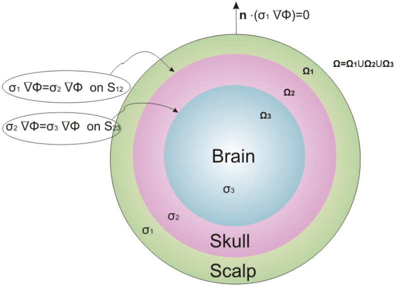

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- Linearity: The body's tissues are passive conductor.

Ill Posedness of the Inverse EEG

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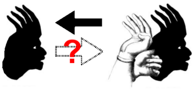

Different Sources' Configurations give the same potential measurements

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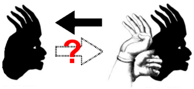

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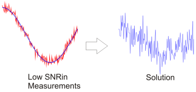

Instability of the solution to the noise in the input data

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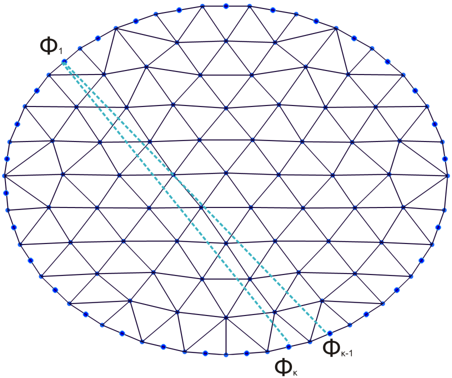
Formulation: Estimate Field **E** employing

$$\text{longitudinal Line Integral: } I_{L_k} = \int_{L_k} \mathbf{E} \cdot \hat{\mathbf{r}} dr = \Phi(\mathbf{a}) - \Phi(\mathbf{b})$$

where L_k is a straight line (so called Ray) which connects points **a** and **b** on the domain's boundaries with potential value Φ and $\hat{\mathbf{r}}$ is the unit vector in the direction of line L_k.

As at an instant:$\mathbf{E} = -\nabla\Phi$ in Ω

Line Integrals Along Straight Lines on a Bounded Domain



By taking all the differences between boundary measurements Φ we estimate the line integrals values.

Numerical Formulation: Ray Projection Matrix

Numerical Formulation of the problem:

$$\boxed{\mathbf{b} \approx \bar{\mathbf{A}}\mathbf{x}}$$

where

- $\mathbf{b}$: differences of the boundary potential measurements;
- $\mathbf{x} = [\mathbf{E}_x|\mathbf{E}_y|\mathbf{E}_y]^T$ the unknown field components at the discrete elements of the domain;
- $\bar{\mathbf{A}}^{m \times n}$ Ray Projection Matrix: the coefficients represent the contribution of the projection of the unknown field components at each discrete element of the domain.

Ray Projection Matrix: Ill Conditioning

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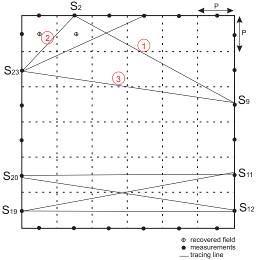
The ill conditioning is also inherently related with the irrotational property of reconstructed field $\mathbf{E}$ (i.e. $\mathbf{E} = -\nabla\Phi$) and the modeling of the problem.

Irrotational Property and Ill-Conditioning

Each continuous Line Integral can be expressed as a linear combination.

recall: $\int_{Closed-Path} \mathbf{E}dc = 0$ as $\mathbf{E} = -\nabla\Phi$



- $\int_{L_1} \mathbf{E} \cdot d\mathbf{r} + \int_{L_2} \mathbf{E} \cdot d\mathbf{r} + \int_{L_3} \mathbf{E} \cdot d\mathbf{r} = 0$
- For N Potential Measurements
- $m = \frac{N(N-1)}{2}$ Integral equations
- $N-1$ independent continues Integral equations in order to avoid loops

  It seems that we have less information than expected!!!

Rank of Ray Projection Matrix

The problem is solved in the discrete domain.
Thus,

- Discretization and interpolation errors are introduced in approximated *Line Integrals*.
- The summation of any set of the approximated equations cannot be zero in any closed path.
- No linear dependencies between equations.
- matrix $\bar{\mathbf{A}}$ of $\mathbf{b} = \bar{\mathbf{A}}\bar{\mathbf{x}}_{LS}$ has full rank and condition number $k(\bar{\mathbf{A}}) < \infty$.

Numerical Solution Error

The solution depends on the discretization error (or the spatial resolution).

If spatial resolution of the domain is finite, we have shown that the relative error (RE)

$$\frac{\|\mathbf{x}_{exact} - \bar{\mathbf{x}}_{LS}\|}{\|\mathbf{x}_{exact}\|} \leq \frac{e_b}{1 - e_b} k(\mathbf{A}) < \infty$$

is bounded.

$0 < e_b < 1$ is the discretization error coefficient. Existence of error bound

ensures invertibility of the discrete problem.

See: Vector Field Tomography: Reconstruction of an Irrotational Field in the Discrete Domain. DOI: 10.2316/P.2012.778-021 Proceeding

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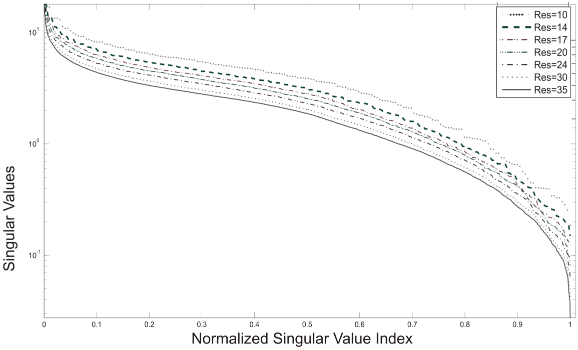
Conditioning of Ray Projection Matrix

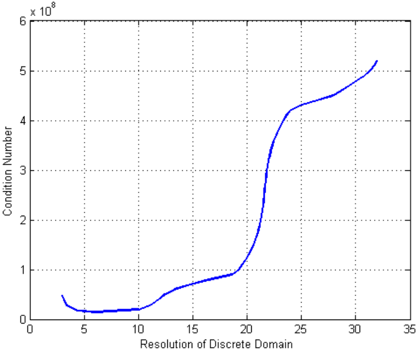
Singular Values: Indicator of ill Conditioning.



→ Rapidly decay of Singular Values to zero ⇒ increase the instability of the linear system.

Conditioning of Ray Projection Matrix

Condition Number:



Condition number k of $\bar{\mathbf{A}}$ $\rightarrow$ transfer error from matrix $\bar{\mathbf{A}}$ and vector $\mathbf{b}$ to the solution $\mathbf{x}$.

Reconstruction of Static Electric Fields

Reconstruction in square domain.

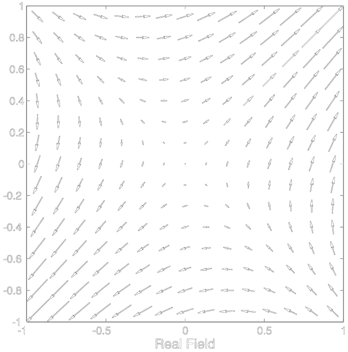


Figure: Actual Field

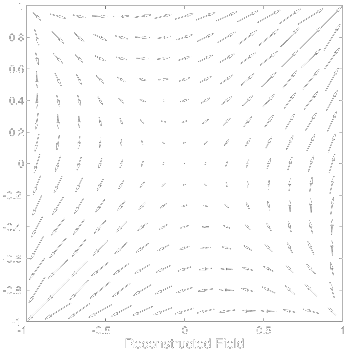
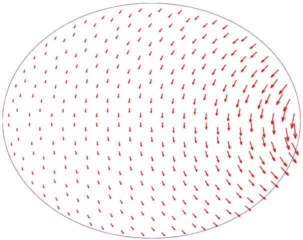


Figure: Reconstructed Field

Reconstruction of Static Fields

Reconstruction in circular domain.



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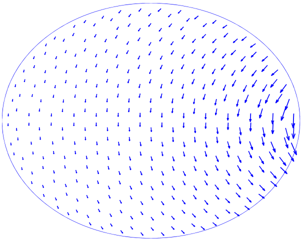

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Comparison with PDE Approach

- Current EEG methods solve the Forward Problem
  i.e. try to adopt a sources' configuration which better describes the recordings.
- In the Line Integral Formulation there is no need for prior information concerning the sources' formulation.
- The boundary conditions are directly incorporated in the line integral formulation.

Future Work

Line Integral Formulation:

- Ray Projection Matrix ill conditioned: further regularization for stable final solution.
- Incomplete data: finite number of sensors along the boundaries What is the most appropriate sensors placements along the boundaries such as to trace the geometry.
- Only a few points can be reconstructed: prior information is needed.
- The reconstruction of the electric field can be used as initial estimate for the sources' localization problem.