Complementary Mean Square Analysis of Augmented CLMS for Second-Order Noncircular Gaussian Signals

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***Abstract*—A novel physical insight is provided into the behavior and performance of the augmented complex least mean square (ACLMS) algorithm for widely linear adaptive estimation of general second-order noncircular (improper) Gaussian signals, whereby the off-diagonal elements of the covariance and complementary covariance matrices are nonzero. This is achieved through a novel complementary mean square analysis, a counterpart to the standard mean square analysis, and which focuses on the behavior of the complementary second-order statistics of the output error and the augmented weight error vector. We next establish the effect of the degree of input noncircularity on the evolution for these two key parameters that govern the ACLMS. Both transient and steady-state performances are addressed and a stability bound on the step-size for their convergence is established. Simulations in the system identification setting support the analysis.**

***Index Terms*—Augmented complex least mean square, complementary mean square analysis, second-order noncircularity.**

I. Introduction

THE design of adaptive filters in the complex domain $\mathbb{C}$ is an important topic in signal processing. It has been widely accepted that the standard, strictly linear, model for complex data is not guaranteed to capture the full second-order statistical relationship between the input and the output, as generic strictly linear extensions of real-valued estimators cater only for second-order circular (proper) data that exhibit rotation invariant probability distributions [1]–[3]. To completely utilize the second-order information available within general complex signals, we therefore have to consider both the standard and complementary second-order statistics, i.e., the covariance and complementary covariance matrices. This is typically achieved through the widely linear estimation framework, based on an augmented input vector that comprises the input vector and its complex conjugate [4]–[6].

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By design, the widely linear estimation framework provides modeling advantages and significant performance gains over its strictly linear counterpart, for second-order noncircular (improper) signals that exhibit a nonzero complementary covariance matrix. Widely linear estimation has inspired the development of the augmented complex least mean square (ACLMS) [7], [8] and its variants [9]–[13], and has found numerous applications in signal processing, communications, power systems, and renewable energy [7], [8], [14]–[18].

While current approaches account for Gaussian and non-Gaussian distributed second-order noncircular input data [11]–[13], [19]–[23], a complete understanding of the behavior of ACLMS is still an open issue. For example, current analyses inherit the principles from the corresponding analyses in the real domain [24]–[27], based on the variance of the output error and the augmented weight error covariance matrix, as well as their transient and steady-state behaviors. We argue that this reflects only one aspect of the full second-order statistics in $\mathbb{C}$, since the evolution of the complementary second-order statistics of these variables has not yet been taken into consideration. It is therefore natural to investigate the effects of input noncircularity on the output error and the augmented weight error vector, the two key system parameters that govern the adaptive filtering algorithms, and to use this knowledge to enhance our understanding of ACLMS.

To this end, we introduce complementary mean square analysis of ACLMS, as a counterpart of the standard mean square analysis, which investigates the evolution of the complementary second-order statistics for improper Gaussian signals. We illustrate that the convergence of the second-order noncircularity of both the output error and the augmented weight error vector is a complement to the recursive minimization of the mean square error (MSE)-based cost function. Theoretical evaluation in both transient and steady-state scenarios is performed, to yield a bound on the step-size, which ensures the converging noncircular behavior of ACLMS.

Notation: Lowercase letters are used to denote scalars, a, boldface letters for column vectors, $\mathbf{a}$, and boldface uppercase letters for matrices, $\mathbf{A}$. The symbols $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$ are, respectively, the transpose, complex conjugate and Hermitian transpose operators. The symbol $E[\cdot]$ denotes the statistical expectation operator, $\mathrm{Tr}[\cdot]$ represents the trace of a matrix, and $\bar{\lambda}[\cdot]$ is the largest eigenvalue of a matrix.

II. Complementary Mean Square Analysis of ACLMS

Consider a second-order noncircular (improper) desired signal $d(k)$ generated by the widely linear model [2], [3]

$$d(k) = \mathbf{h}^{\mathrm{o}H}\mathbf{x}(k) + \mathbf{g}^{\mathrm{o}H}\mathbf{x}^*(k) + q(k) = \mathbf{w}^{\mathrm{o}H}\mathbf{z}(k) + q(k) \quad (1)$$

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where $\mathbf{h}^{\mathrm{o}}$ and $\mathbf{g}^{\mathrm{o}}$ denote the optimal standard and conjugate system impulse response vectors of length L, $\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_L(k)]^T$ is the tap input vector to the unknown system at time instant k, and $q(k)$ is an i.i.d. measurement noise, which, for generality, is assumed to be a zero-mean improper process with variance $E[q(k)q^*(k)] = \sigma_q^2$ and complementary variance $E[q^2(k)] = \widetilde{\sigma}_q^2$. For compactness, the widely linear model can also be represented in an augmented form, where $\mathbf{z}(k) = [\mathbf{x}^T(k), \mathbf{x}^H(k)]^T \in \mathbb{C}^{2L\times 1}$ is the augmented input vector and $\mathbf{w}^{\mathrm{o}} = [\mathbf{h}^{\mathrm{o}}, \mathbf{g}^{\mathrm{o}}]^T \in \mathbb{C}^{2L\times 1}$ the augmented true system impulse vector. For generality, we further assume that the input $\mathbf{x}(k)$ is a correlated second-order noncircular Gaussian process, i.e., the off-diagonal elements in both the covariance matrix $\mathbf{R} = E[\mathbf{x}(k)\mathbf{x}^H(k)]$ and the complementary covariance matrix $\mathbf{P} = E[\mathbf{x}(k)\mathbf{x}^T(k)]$ do exist.

A. Complementary Mean Square Convergence of ACLMS

The ACLMS estimates the set of system parameters, $\mathbf{w}^{\mathrm{o}}$, by minimizing the MSE cost function, $J(k) = E[e(k)e^*(k)]$, where the output error is $e(k) = d(k) - \mathbf{w}^H(k)\mathbf{z}(k)$. The augmented weight update equation then becomes

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu e^*(k)\mathbf{z}(k) \tag{2}$$

where the parameter $\mu \in \mathbb{R}^+$ is the step-size. By defining the augmented weight error vector $\widetilde{\mathbf{w}}(k)$ as $\widetilde{\mathbf{w}}(k) = \mathbf{w}^{\mathrm{o}} - \mathbf{w}(k)$, the error $e(k)$ and the recursion for $\widetilde{\mathbf{w}}(k)$ can be rewritten as

$$e(k) = \widetilde{\mathbf{w}}^H(k)\mathbf{z}(k) + q(k) \tag{3}$$

$$\widetilde{\mathbf{w}}(k+1) = \left(\mathbf{I} - \mu\mathbf{z}(k)\mathbf{z}^H(k)\right)\widetilde{\mathbf{w}}(k) - \mu q^*(k)\mathbf{z}(k). \tag{4}$$

The standard mean square performance analysis of ACLMS follows the analysis from the real LMS and investigates the convergence of the MSE, $J(k) = E[|e(k)|^2]$, and the augmented weight error covariance matrix, $\mathbf{K}(k) = E[\widetilde{\mathbf{w}}(k)\widetilde{\mathbf{w}}^H(k)]$ [19]–[23]. However, since in general the Gaussian input $\mathbf{x}(k)$ and the system noise $q(k)$ are both improper, it is natural to investigate how the so introduced noncircularity propagates into $e(k)$ and $\widetilde{\mathbf{w}}(k)$. To this end, we first define the complementary mean square error (CMSE), $\widetilde{J}(k)$, which maintains the information about the second-order noncircularity of $e(k)$, and is given by

$$\widetilde{J}(k) = E[e^2(k)]. \tag{5}$$

Upon employing the standard independent assumptions, that is, the noise $q(k)$ is statistically independent of any other signal in ACLMS and $\widetilde{\mathbf{w}}(k)$ is statistically independent of the augmented input $\mathbf{z}(k)$ [19]–[23], $\widetilde{J}(k)$ can be evaluated as

$$\begin{aligned}\widetilde{J}(k) &= E[q^2(k)] + E\left[\widetilde{\mathbf{w}}^H(k)\mathbf{z}(k)\mathbf{z}^T(k)\widetilde{\mathbf{w}}^*(k)\right] \\ &= \widetilde{\sigma}_q^2 + \mathrm{Tr}\left[\mathbf{P}^a\widetilde{\mathbf{K}}^*(k)\right]\end{aligned} \tag{6}$$

where $\mathbf{P}^a = E[\mathbf{z}(k)\mathbf{z}^T(k)]$ is the augmented complementary covariance matrix of the input $\mathbf{x}(k)$, defined as [21]

$$\begin{aligned}\mathbf{P}^a &= E[\mathbf{z}(k)\mathbf{z}^T(k)] \\ &= \begin{bmatrix} E[\mathbf{x}(k)\mathbf{x}^T(k)] & E[\mathbf{x}(k)\mathbf{x}^H(k)] \\ E[\mathbf{x}^*(k)\mathbf{x}^T(k)] & E[\mathbf{x}^*(k)\mathbf{x}^H(k)] \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{R} \\ \mathbf{R}^* & \mathbf{P}^* \end{bmatrix}\end{aligned} \tag{7}$$

and

$$\widetilde{\mathbf{K}}(k) = E[\widetilde{\mathbf{w}}(k)\widetilde{\mathbf{w}}^T(k)] \tag{8}$$

is the complementary covariance matrix of the augmented weight error vector $\widetilde{\mathbf{w}}(k)$.

Remark 1: As indicated by (6), the second-order noncircularity of $e(k)$ arises from two sources: 1) a constant term $\widetilde{\sigma}_q^2$; 2) a time-variant term depending on $\widetilde{\mathbf{K}}(k)$.

Upon multiplying both sides of (4) with $\widetilde{\mathbf{w}}^T(k+1)$, and again applying the statistical expectation operator and the standard independence assumptions, we obtain

$$\begin{aligned}\widetilde{\mathbf{K}}(k+1) &= \widetilde{\mathbf{K}}(k) - \mu\mathbf{R}^a\widetilde{\mathbf{K}}(k) - \mu\widetilde{\mathbf{K}}(k)\mathbf{R}^{a*} + \mu^2\widetilde{\sigma}_q^{2*}\mathbf{P}^a \\ &+ \mu^2 E\left[\mathbf{z}(k)\mathbf{z}^H(k)\widetilde{\mathbf{w}}(k)\widetilde{\mathbf{w}}^T(k)\mathbf{z}^*(k)\mathbf{z}^T(k)\right]\end{aligned} \tag{9}$$

where

$$\begin{aligned}\mathbf{R}^a &= E[\mathbf{z}(k)\mathbf{z}^H(k)] \\ &= \begin{bmatrix} E[\mathbf{x}(k)\mathbf{x}^H(k)] & E[\mathbf{x}(k)\mathbf{x}^T(k)] \\ E[\mathbf{x}^*(k)\mathbf{x}^H(k)] & E[\mathbf{x}^*(k)\mathbf{x}^T(k)] \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{P} \\ \mathbf{P}^* & \mathbf{R}^* \end{bmatrix}\end{aligned} \tag{10}$$

is the augmented covariance matrix of the input $\mathbf{x}(k)$ [2], [3]. The (i,j)th entry of the statistical expectation of the last term on the right-hand side (RHS) of (9) can be expressed as

$$\begin{aligned}&\left\{E\left[\mathbf{z}(k)\mathbf{z}^H(k)\widetilde{\mathbf{w}}(k)\widetilde{\mathbf{w}}^T(k)\mathbf{z}^*(k)\mathbf{z}^T(k)\right]\right\}_{ij} \\ &= \sum_{l=1}^{2L}\sum_{m=1}^{2L} E\left[z_i(k)z_l^*(k)z_m^*(k)z_j(k)\right]E\left[\widetilde{w}_l(k)\widetilde{w}_m(k)\right].\end{aligned} \tag{11}$$

From the Gaussian fourth-order moment factorizing theorem,

$$E\left[z_i(k)z_l^*(k)z_m^*(k)z_j(k)\right] = r_{il}^a r_{jm}^a + r_{im}^a r_{jl}^a + p_{ij}^a p_{lm}^{a*} \tag{12}$$

and hence,

$$\begin{aligned}&E\left[\mathbf{z}(k)\mathbf{z}^H(k)\widetilde{\mathbf{w}}(k)\widetilde{\mathbf{w}}^T(k)\mathbf{z}^*(k)\mathbf{z}^T(k)\right] \\ &= 2\mathbf{R}^a\widetilde{\mathbf{K}}(k)\mathbf{R}^{a*} + \mathbf{P}^a\,\mathrm{Tr}\left[\mathbf{P}^{a*}\widetilde{\mathbf{K}}(k)\right].\end{aligned} \tag{13}$$

Therefore, the recursion for the complementary covariance matrix of $\widetilde{\mathbf{w}}(k)$, that is, $\widetilde{\mathbf{K}}(k)$ in (8), now becomes

$$\begin{aligned}\widetilde{\mathbf{K}}(k+1) &= \widetilde{\mathbf{K}}(k) - \mu\left(\mathbf{R}^a\widetilde{\mathbf{K}}(k) + \widetilde{\mathbf{K}}(k)\mathbf{R}^{a*}\right) \\ &+ \mu^2\left(2\mathbf{R}^a\widetilde{\mathbf{K}}(k)\mathbf{R}^{a*} + \mathbf{P}^a\,\mathrm{Tr}\left[\mathbf{P}^{a*}\widetilde{\mathbf{K}}(k)\right] + \widetilde{\sigma}_q^{2*}\mathbf{P}^a\right).\end{aligned} \tag{14}$$

Remark 2: Equations (6) and (14) indicate that the second-order noncircularity of both $e(k)$ and $\widetilde{\mathbf{w}}(k)$ arises from the second-order noncircularity of the Gaussian input $\mathbf{x}(k)$ and/or the system noise $q(k)$; together (6) and (14) fully describe the complementary mean square performance of ACLMS.

B. Complementary Mean Square Stability of ACLMS

We now consider the sufficient condition on the step-size μ for the convergence of $\widetilde{\mathbf{K}}(k)$, which would also guarantee the convergence of the CMSE $\widetilde{J}(k)$ in (6). This is challenging in the sense that it requires a simultaneous diagonalization of both the augmented covariance matrix, $\mathbf{R}^a$, and its complementary counterpart $\mathbf{P}^a$. Following on the recent work on the standard mean square analysis of ACLMS in [21], in order to exploit the special structures of $\mathbf{R}^a$ and $\mathbf{P}^a$, we here employ the recently introduced approximate uncorrelating transform (AUT) [28], which allows for a joint direct diagonalization of both $\mathbf{R}$ and $\mathbf{P}$ based on a single singular value decomposition (SVD), within some reasonable approximations.

The AUT states that the unitary matrix $\mathbf{Q}$, which is used in the Takagi factorization [29] to diagonalize the complex-valued symmetric complementary covariance matrix $\mathbf{P}$, can also be used to approximately diagonalize the covariance matrix $\mathbf{R}$, so that

$$\mathbf{P} = \mathbf{Q}\boldsymbol{\Lambda}_\mathbf{p}\mathbf{Q}^T, \qquad \mathbf{R} \simeq \mathbf{Q}\boldsymbol{\Lambda}_\mathbf{r}\mathbf{Q}^H \tag{15}$$

where $\mathbf{Q}\mathbf{Q}^H = \mathbf{I}$, $\boldsymbol{\Lambda}_\mathbf{p} = \text{diag}\{\lambda_1^p, \lambda_2^p, \ldots, \lambda_L^p\}$ is a diagonal matrix of real-valued entries, $\lambda_1^p \geq \lambda_2^p \geq \cdots \geq \lambda_L^p$ are the nonnegative square roots of $\mathbf{P}\mathbf{P}^H$ [29], and $\boldsymbol{\Lambda}_\mathbf{r} = \text{diag}\{\lambda_1^r, \lambda_2^r, \ldots, \lambda_L^r\}$, $\lambda_1^r \geq \lambda_2^r \geq \cdots \geq \lambda_L^r$, with λ_l^r being the real-valued eigenvalues of $\mathbf{R}$. The approximation in (15) is valid for univariate data, and the equality is achieved when $\mathbf{x}(k)$ is real-valued, i.e., maximum noncircular [21], [28]. From (15), $\mathbf{R}^a$ and $\mathbf{P}^a$ can now be jointly diagonalized as [21]

$$\mathbf{R}^a \simeq \bar{\mathbf{Q}}\mathbf{D}_\mathbf{r}\bar{\mathbf{Q}}^H, \qquad \mathbf{P}^a = \bar{\mathbf{Q}}\mathbf{D}_\mathbf{p}\bar{\mathbf{Q}}^T \tag{16}$$

where the unitary matrix $\bar{\mathbf{Q}}$ has the form

$$\bar{\mathbf{Q}} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{Q} & -\mathbf{Q} \\ \mathbf{Q}^* & \mathbf{Q}^* \end{bmatrix}. \tag{17}$$

The diagonal matrices $\mathbf{D}_\mathbf{r}$ and $\mathbf{D}_\mathbf{p}$ are identical except for the opposite signs of the last L diagonal elements, that is

$$\mathbf{D}_\mathbf{r} = \begin{bmatrix} \boldsymbol{\Lambda}_\mathbf{r} + \boldsymbol{\Lambda}_\mathbf{p} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_\mathbf{r} - \boldsymbol{\Lambda}_\mathbf{p} \end{bmatrix} \tag{18}$$

$$\mathbf{D}_\mathbf{p} = \begin{bmatrix} \boldsymbol{\Lambda}_\mathbf{r} + \boldsymbol{\Lambda}_\mathbf{p} & \mathbf{0} \\ \mathbf{0} & -(\boldsymbol{\Lambda}_\mathbf{r} - \boldsymbol{\Lambda}_\mathbf{p}) \end{bmatrix}. \tag{19}$$

We can now rotate the augmented input vector, $\mathbf{z}(k)$, and the augmented weight error vector, $\widetilde{\mathbf{w}}(k)$, to give

$$\hat{\mathbf{z}}(k) = \bar{\mathbf{Q}}^H\mathbf{z}(k), \qquad \hat{\mathbf{w}}(k) = \bar{\mathbf{Q}}^H\widetilde{\mathbf{w}}(k) \tag{20}$$

and decompose the term $\text{Tr}[\mathbf{P}^{a*}\widetilde{\mathbf{K}}(k)]$ on the RHS of (14) as

$$\begin{aligned} \text{Tr}\left[\mathbf{P}^{a*}\widetilde{\mathbf{K}}(k)\right] &= E\left[\widetilde{\mathbf{w}}^T(k)\mathbf{z}^*(k)\mathbf{z}^H(k)\widetilde{\mathbf{w}}(k)\right] \\ &= E\left[\hat{\mathbf{w}}^T(k)\mathbf{D}_\mathbf{p}\hat{\mathbf{w}}(k)\right] = \mathbf{p}^T\widetilde{\boldsymbol{\kappa}}(k) \end{aligned} \tag{21}$$

where $\mathbf{p} = [p_1, p_2, \ldots, p_{2L}]$ contains the diagonal elements of $\mathbf{D}_\mathbf{p}$, given by

$$p_l = \begin{cases} \lambda_l^r + \lambda_l^p, & l = 1, 2, \ldots, L \\ -(\lambda_{l-L}^r - \lambda_{l-L}^p), & l = L+1, L+2, \ldots, 2L \end{cases} \tag{22}$$

while the vector $\widetilde{\boldsymbol{\kappa}}(k)$ contains the augmented weight error complementary variances after input data diagonalization, the components of which are the diagonal elements of $E[\hat{\mathbf{w}}(k)\hat{\mathbf{w}}^T(k)]$. Therefore, based on (6) and (21), we have

$$\widetilde{J}(k) = \widetilde{\sigma}_q^2 + \mathbf{p}^T\widetilde{\boldsymbol{\kappa}}^*(k) \tag{23}$$

from which, based on (14), (16), (20), and (21), and after a few algebraic manipulations, the evolution of $\widetilde{\boldsymbol{\kappa}}(k)$ can be expressed as

$$\widetilde{\boldsymbol{\kappa}}(k+1) = \underbrace{\left[(\mathbf{I} - \mu\mathbf{D}_\mathbf{r})^2 + \mu^2\mathbf{D}_\mathbf{r}^2 + \mu^2\mathbf{p}\mathbf{p}^T\right]}_{\mathbf{A}}\widetilde{\boldsymbol{\kappa}}(k) + \mu^2\widetilde{\sigma}_q^{2*}\mathbf{p}. \tag{24}$$

Remark 3: Strictly speaking, since the complex domain $\mathbb{C}$ is not ordered, convergence based on complex-valued quantities should be treated with caution. However, the real-valued nature of the evolution matrix $\mathbf{A}$ guarantees that the real and imaginary parts of $\widetilde{\boldsymbol{\kappa}}(k)$ evolve independently, which enables us to treat the convergence of $\widetilde{\boldsymbol{\kappa}}(k)$ through a separate evolution of its real and imaginary parts. This is also the case with the CMSE, $\widetilde{J}(k)$, whose convergence depends on that of $\widetilde{\boldsymbol{\kappa}}(k)$, as indicated by (23).

In order for the recursion in (24) to converge, the eigenvalues of $\mathbf{A}$ have to be less than unity. Applying the Weyl inequality, which states that for two Hermitian matrices $\mathbf{X}$ and $\mathbf{Y}$, $\bar{\lambda}[\mathbf{X} + \mathbf{Y}] \leq \bar{\lambda}[\mathbf{X}] + \bar{\lambda}[\mathbf{Y}]$, to $\mathbf{A}$ in (24) yields

$$\bar{\lambda}[\mathbf{A}] \leq \bar{\lambda}\left[(\mathbf{I} - \mu\mathbf{D}_\mathbf{r})^2\right] + \mu^2\left(\bar{\lambda}\left[\mathbf{D}_\mathbf{r}^2\right] + \bar{\lambda}[\mathbf{p}\mathbf{p}^T]\right). \tag{25}$$

Since $\bar{\lambda}[\mathbf{p}\mathbf{p}^T] = \text{Tr}[\mathbf{D}_\mathbf{p}^2] = \text{Tr}[\mathbf{D}_\mathbf{r}^2] = \bar{\lambda}[\mathbf{r}\mathbf{r}^T]$, where $\mathbf{r} = [r_1, r_2, \ldots, r_{2L}]^T$ contains the diagonal elements of $\mathbf{D}_\mathbf{r}$,

$$r_l = \begin{cases} \lambda_l^r + \lambda_l^p, \ l = 1, 2, \ldots, L \\ \lambda_{l-L}^r - \lambda_{l-L}^p, \ l = L+1, L+2, \ldots, 2L \end{cases} \tag{26}$$

the upper bound on the RHS of (25) is identical to that [21, eq. (14)] in the standard boundedness analysis on μ for the convergence of the MSE, $J(k)$, and the augmented weight error covariance matrix, $\mathbf{K}(k)$, leading to a unified mean square stability bound on μ, given by [21]

$$0 < \mu < \frac{r_{\min}}{(L+1)r_{\max}^2} \tag{27}$$

where $r_{\min}$ and $r_{\max}$ are, respectively, the minimum and the maximum of r_l in (26).

Remark 4: The unified stability bound on μ in (27), which guarantees both the convergence of the MSE $J(k)$ and the CMSE $\widetilde{J}(k)$ of ACLMS, indicates that any μ within this conservative bound which enables the recursive minimization of $J(k)$ also recursively minimizes the second-order noncircularity of $e(k)$, represented by $\widetilde{J}(k)$, in terms of its real and imaginary components separately. Since $\widetilde{J}(k) = E[e^2(k)] = E[e_r^2(k)] - E[e_\imath^2(k)] + 2\imath E[e_r(k)e_\imath(k)]$, this means that both the power difference between the real and imaginary components of $e(k)$, as well as their cross-correlation, are recursively minimized. In other words, the output error $e(k)$ becomes increasingly more second-order circular during the recursive minimization process of $J(k)$, and this is also the case with the augmented weight error vector $\widetilde{\mathbf{w}}(k)$.

C. Steady-State CMSE of ACLMS

Suppose μ is chosen such that the mean square stability of ACLMS is guaranteed, and consider $k \to \infty$ in (23), then the steady-state CMSE $\widetilde{J}(\infty)$ can be evaluated as

$$\widetilde{J}(\infty) = \widetilde{\sigma}_q^2 + \mathbf{p}^T\widetilde{\boldsymbol{\kappa}}^*(\infty) \tag{28}$$

where, based on (24), $\widetilde{\boldsymbol{\kappa}}^*(\infty)$ can be derived as

$$\widetilde{\boldsymbol{\kappa}}^*(\infty) = \left[\underbrace{2\mathbf{D}_\mathbf{r} - 2\mu\mathbf{D}_\mathbf{r}^2}_{\mathbf{B}} - \mu\mathbf{p}\mathbf{p}^T\right]^{-1}\mu\widetilde{\sigma}_q^2\mathbf{p}. \tag{29}$$

By employing the matrix inversion lemma, and after a few algebraic manipulations, we have

$$\widetilde{\boldsymbol{\kappa}}^*(\infty) = \left[\mathbf{I} + \frac{\mu\mathbf{B}^{-1}\mathbf{p}\mathbf{p}^T}{1 - \mu\mathbf{p}^T\mathbf{B}^{-1}\mathbf{p}}\right]\mu\widetilde{\sigma}_q^2\mathbf{B}^{-1}\mathbf{p}. \tag{30}$$

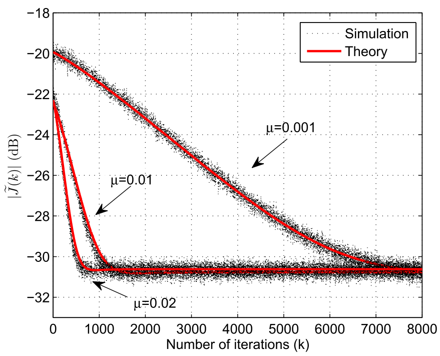

Fig. 1. Learning curves of the absolute value of CMSE, $|\widetilde{J}(k)|$, of ACLMS for a correlated second-order noncircular AR(1) Gaussian input $x(k)$, where $x(k) = 0.7x(k-1) + u(k)$. The driving noise $u(k)$ was a white second-order noncircular Gaussian process with $\sigma_u^2 = 1$ and $\widetilde{\sigma}_u^2 = 0.9$, while the variance of the system noise, $q(k)$, was $\sigma_q^2 = 0.001$.

Upon substituting (30) into (28), we arrive at

$$\widetilde{J}(\infty) = \widetilde{\sigma}_q^2 + \frac{\mu\widetilde{\sigma}_q^2 \mathbf{p}^T \mathbf{B}^{-1}\mathbf{p}}{1-\mu\mathbf{p}^T\mathbf{B}^{-1}\mathbf{p}}$$

$$= \widetilde{\sigma}_q^2 + \frac{\sum_{l=1}^{2L} \frac{\mu\widetilde{\sigma}_q^2 r_l}{2-2\mu r_l}}{1-\sum_{l=1}^{2L} \frac{\mu r_l}{2-2\mu r_l}}. \quad (31)$$

The last step of (31) is achieved by considering the fact that $p_l^2 = r_l^2$ for $\forall l$. Physically, after ACLMS converges, the steady-state CMSE, $\widetilde{J}(\infty)$, in (31) serves as the lower bound on the degree of second-order noncircularity of the output error $e(k)$.

III. Simulations

Numerical experiments in a system identification setting described in (1) were conducted to evaluate the theoretical findings. The system coefficients to be identified, $\mathbf{h}^o$ and $\mathbf{g}^o$, were those typically used for modeling frequency-dependent inphase/quadrature (I/Q) imbalance in the direct-conversion receivers for wide-band wireless systems, given by [30], [31]

$$\mathbf{h}^o = \frac{\mathbf{a} + \gamma e^{-\jmath\theta}\mathbf{b}}{2}, \quad \mathbf{g}^o = \frac{\mathbf{a} - \gamma e^{\jmath\theta}\mathbf{b}}{2} \quad (32)$$

where $\mathbf{a} = [0.01, 1, 0.01]^T$ and $\mathbf{b} = [0.01, 1, 0.2]^T$ are, respectively, the low-pass filters for the I- (real) and Q- (imaginary) branches of the receiver, and the gain mismatch and phase mismatch between the two branches are $\gamma = 1.03$ and $\theta = 3^\circ$ [31]. All the simulation results were obtained by averaging over 10,000 independent trials.

We first investigated the validity of the proposed complementary mean square convergence analysis of the ACLMS algorithm. As discussed in Remark 3, the complementary MSE (CMSE) $\widetilde{J}(k) = E[e^2(k)]$ and the complementary covariance matrix $\widetilde{\mathbf{K}}(k) = E[\widetilde{\mathbf{w}}(k)\widetilde{\mathbf{w}}^T(k)]$ are complex-valued, and their convergence is governed by the individual convergence of their real parts and imaginary parts. Fig. 1 shows the theoretical and simulated convergence behaviors of $\widetilde{J}(k)$, measured by its absolute value $|\widetilde{J}(k)|$, when applying ACLMS for a second-order noncircular autoregressive (AR)(1) Gaussian input and different step-sizes μ. The theoretical CMSE, $\widetilde{J}(k)$, was evaluated based on (6), where $\widetilde{\mathbf{K}}(k)$ was recursively obtained by (14). For this experiment, the upper bound on μ, numerically evaluated by (27), to guarantee their convergence as well as that of the standard MSE $J(k)$, was 0.0226. It can be observed from Fig. 1 that our theoretical analysis accurately described the evolution of $\widetilde{J}(k)$, and implicitly that of $\widetilde{\mathbf{K}}(k)$, in both transient and steady-state stages. This experiment also conforms with the analysis in Remark 4, which states that the recursive minimization process of $J(k)$ also minimizes the degree of second-order noncircularity of the output error $e(k)$ and the augmented weight error vector $\widetilde{\mathbf{w}}(k)$ of the ACLMS algorithm.

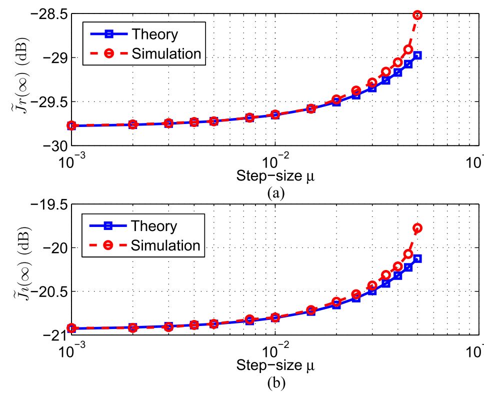

Fig. 2. Steady-state CMSE, $\widetilde{J}(\infty)$, evaluated through its real and imaginary components, of ACLMS for a correlated second-order noncircular AR(4) Gaussian input $x(k)$, where $x(k) = 1.79x(k-1) - 1.85x(k-2) + 1.27x(k-3) - 0.41x(k-4) + u(k)$. The driving noise $u(k)$ was a white second-order noncircular Gaussian process with $\sigma_u^2 = 1$ and $\widetilde{\sigma}_u^2 = 0.7$, while the variance of the system noise, $q(k)$, was $\sigma_q^2 = 0.01$. (a) Real part $\widetilde{J}_r(\infty)$, and (b) imaginary part $\widetilde{J}_\imath(\infty)$.

In order to achieve the analytical evaluation of the steady-state CMSE, $\widetilde{J}(\infty)$, of ACLMS, whose closed-form expression is given in (31), the AUT was used to diagonalize both the covariance and complementary covariance matrices, $\mathbf{R}$ and $\mathbf{P}$, with a single SVD. Fig. 2 shows the theoretical and simulated steady-state CMSE, $\widetilde{J}(\infty)$, represented through its real and imaginary components, $\widetilde{J}_r(\infty)$ and $\widetilde{J}_\imath(\infty)$, when applying ACLMS for a second-order noncircular AR(4) process. The step-size μ varied from 0.001 to 0.05, and this range guaranteed the stability bound in (27). The empirical results were more closely matched with the analytical ones for smaller step-sizes as compared with larger step-sizes, where a breakdown of the assumption that the augmented weight error vector $\widetilde{\mathbf{w}}(k)$ is statistically independent of $\mathbf{x}(k)$ may occur.

IV. Conclusion

We have introduced a novel complementary mean square performance analysis of the ACLMS algorithm for second-order noncircular Gaussian signals. As compared with the standard mean square analysis, our analysis has offered a new dimension which is physically more intuitive regarding the propagation of the noncircularity of both the system input and noise within ACLMS. This has been achieved by investigating the convergence of the degree of second-order noncircularity of the output error and the augmented weight error vector in both transient and steady-state stages. The analysis has established that the recursive minimization of the standard mean square error, which is the basis for the convergence of ACLMS, also minimizes the second-order noncircularity of the output error and the augmented weight error vector. Simulations in a system identification setting support the analysis.

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