The Correlation Preserving Transform Resolves Phase Ambiguity in Complex Decorrelating Transforms

C. Cheong Took
Department of Computing
University of Surrey UK
Email: c.cheongtook@surrey.ac.uk

S. C. Douglas
Department of Electrical Engineering
Southern Methodist University USA
Email: douglas@engr.smu.edu

D. P. Mandic
Department of Electronic Engineering
Imperial College London UK
Email: d.mandic@ic.ac.uk

***Abstract*—A solution to the problem of intraference is obtained by recognising that the current diagonalisation schemes for the complex symmetric pseudocovariance matrix are not adequate to preserve its intrinsic complex-valued nature. To this end, we propose the correlation preserving transform (CPT), which both maintains the degree of the intrinsic correlation within a recovered bivariate source (minimum intraference) and minimises the correlation between different bivariate sources (minimum interference). Unlike the existing literature, this work therefore considers both the correlation structure between the data channels within bivariate sources and inter-source correlation. The efficacy of the proposed method is validated through analysis and simulations.**

I. Introduction

The recently introduced concept of intraference [1] is a crucial yet largely overlooked phenomenon arising in complex-valued signal processing, whereby phase information is lost when using uncorrelating transforms. This loss of phase information is inherent in all current matrix decorrelation techniques, and occurs when the pseudocovariance $p = E\{x^2\}$ of a noncircular complex-valued signal is forced to be real-valued, although in reality it is not. To illustrate this problem, consider the Euler representation of a noncircular signal $x = |x|e^{\imath\theta}$ whose pseudocovariance is also complex valued

$$p = E\{x^2\} = E\{|x|^2 e^{\imath 2\theta}\} \qquad (1)$$

If the phase information θ is a multiple of π, then the intraference becomes the trivial case of phase ambiguity in independent component analysis (ICA), which is also referred to as the sign ambiguity [2]. However, if the phase information $\theta \neq k\pi, \quad \forall\, k$ and the estimated source is forced to have a real-valued pseudocovariance, then this loss in phase information leads to a poor estimate of the actual source. The intraference therefore refers to the level of error arising from an inadequate estimate of the complex-valued pseudocovariance, and is given by

$$\textbf{Intraference} : E\{\Phi^2\} \quad = \quad E\{|\tilde{\xi}_P|^2\}/|p|^2 \qquad (2)$$

where $\tilde{\xi}_P$ is related to the error in estimating the complex-valued pseudocovariance p; for more detail, see [1]. Observe that a complex-valued pseudocovariance implies that the real (x_R) and imaginary (x_I) part of a complex signal $x = x_R + \imath x_I$ are correlated, since

$$p = E\{x^2\} = E\{x_R^2\} - E\{x_I^2\} + 2\imath \underbrace{E\{x_R x_I\}}_{\text{correlation}} \qquad (3)$$

Remark#1: Current algebraic complex ICA techniques decorrelate both complex sources and their real and imaginary channels, and therefore to preserve source integrity and mitigate the effect of intraference, the correlation between the real and imaginary part of a signal needs to be preserved.

Intraference becomes particularly challenging to tackle when

- estimating *several* noncircular sources simultaneously, as the intraferences for each source accumulate;
- the criterion used to recover the sources simultaneously is in *direct contrast* to the criterion employed to correctly estimate the pseudocovariances of individual sources.

In other words, intraference is particularly difficult to solve when performing ICA between the different sources while at the same time preserving the intrinsic correlation structure within each complex-valued source, a subject of this work. Following on our earlier work on intraference [1], we propose the correlation preserving transform (CPT) to address both blind source separation and intraference while preserving the physical meaning and integrity of the reconstructed sources.

In this work, we first illustrate how current matrix decompositions are not adequate to preserve the complex-valued nature of the pseudocovariance matrix $\mathbf{P} = E\{\mathbf{x}\mathbf{x}^T\}$. More specifically, we show that the phase information is lost in the formulation of the Takagi's factorisation from the singular value decomposition [3]. Takagi's factorisation is required for the diagonalisation of the symmetric pseudocovariance matrix and is used in the strong uncorrelating transform (SUT) for ICA [4] [5]. We therefore consider the SUT as a starting point and demonstrate how it can overcome intraference when enhanced with the proposed correlation preserving transform (CPT). For convenience, the analysis is validated through simulations, motivated by those shown in [1].

II. THE CORRELATION PRESERVING TRANSFORM

The classic result in [3], namely Takagi's factorisation was adopted into signal processing through the work of De Lauthawer *et al.* [4] and later independently by Eriksson *et al.* [5] in the form of the strong uncorrelating transform (SUT) [6]. Takagi's factorisation of a complex-valued symmetric matrix $\mathbf{A} = \mathbf{A}^T$ is a special case of singular value decomposition, and can be computed as

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T \tag{4}$$

where $\mathbf{Q}$ is a unitary matrix, and $\mathbf{\Lambda}$ is a diagonal matrix containing the singular values. The key to the proposed CPT solution is to show that the unitary matrices $\mathbf{U}$ and $\mathbf{V}$ within the singular value decomposition $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^H$ can be expressed

$$\mathbf{V}^* = \mathbf{U}\mathbf{\Delta} \tag{5}$$

where $\mathbf{\Delta} = \mathrm{diag}(e^{\imath\theta_1}, \dots, e^{\imath\theta_n})$, $(\cdot)^*$ denotes the complex conjugation operation, and $(\cdot)^H$ denotes the Hermitian transpose operator. The relation between the right and left eigenvector always holds for distinct singular values, however, when some of the singular values coincide this relationship may not hold [7]; also see (p. 411 [3]). For simplicity, it is generally assumed that for a symmetric matrix $\mathbf{A} = \mathbf{A}^T$, $\mathbf{\Delta} = \mathbf{I}$ such that $\mathbf{U} = \mathbf{V}^*$, however, the matrix $\mathbf{\Delta}$ is not unique and may also depend on the numerical implementation[1] of the singular value decomposition.

For rigour, we shall derive the Takagi's factorisation by exploiting the relationship in (5) as

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\mathbf{S}\mathbf{V}^H = \mathbf{U}\mathbf{S}\mathbf{\Delta}\mathbf{U}^T && (6)\\ &= = \mathbf{U}\mathbf{\Delta}^{1/2}\mathbf{S}\mathbf{\Delta}^{1/2}\mathbf{U}^T = \mathbf{Q}\mathbf{S}\mathbf{Q}^T && (7) \end{aligned}$$

Remark#2: The phase information in (6) and (7) is lost, as using the Takagi's factorisation, the diagonalised pseudocovariance is inherently real-valued. This can be shown through the steps in the computation of Takagi's factorisation, from the general singular value decomposition in (6) to the standard form of Takagi's factorisation in (7), whereby $\mathbf{\Delta}$ is absorbed into $\mathbf{U}$, thus causing loss in phase information in order to satisfy the Takagi's form $\mathbf{P} = \mathbf{Q}\mathbf{S}\mathbf{Q}^T$.

Remark#3: As $\mathbf{Q}^H\mathbf{P}\mathbf{Q}^* = \mathbf{S} \in \mathbb{R}^{N\times N}$, we need to address the problem of calculating the conjugate eigenvectors (con-eigenvectors), which satisfy $\mathbf{A}\mathbf{U}^* = \mathbf{U}\mathbf{D}$. Con-eigenanalysis is, however, non-unique; for example, if μ is a non-negative con-eigenvalue of $\mathbf{A}$ then so too is $\mu\exp(\imath\theta)$, for any θ [3].

A. The Strong Uncorrelating Transform

A simple yet effective tool for complex-valued independent component analysis is the strong uncorrelating transform (SUT) [4], [5], [8], [9]. As its name suggest, the SUT is derived based on the strong assumption of uncorrelatedness, meaning that there is no room to model correlation between the real and imaginary part of an independent component (IC). As such, SUT is not suitable to estimate complex-valued sources which exhibit intrinsic correlation which is reflected in a complex-valued pseudocovariance, a typical case in practice. To achieve statistical independence of the sources, SUT first diagonalises the complex-valued covariance matrix $\mathbf{C}$, then it diagonalises the resulting normalised[2] pseudocovariance $\mathbf{P}$ to a real-valued matrix by using Takagi's factorisation. The following diagonalising steps summarise SUT:

$$\mathbf{\Lambda}_C = \boldsymbol{W}\mathbf{C}\boldsymbol{W}^H = \mathbf{I} \qquad \mathbf{\Lambda}_P = \boldsymbol{W}\mathbf{P}\boldsymbol{W}^T \tag{8}$$

where the symbols $\mathbf{\Lambda}_C$ and $\mathbf{\Lambda}_P$ denote respectively the real-valued diagonalised covariance and pseudocovariance matrices, and $\mathbf{I}$ and $\mathbf{W}$ are respectively the identity matrix and the strong uncorrelating transform, given by

$$\boldsymbol{W} = \mathbf{Q}_P^H\mathbf{C}^{-\frac{1}{2}} \tag{9}$$

The unitary matrix $\mathbf{Q}_P$ can be calculated from the Takagi's factorisation of the pseudocovariance matrix.

B. The Proposed Correlation Preserving Transform

Recall that SUT decorrelates multivariate complex data in two steps: (i) by performing decorrelation between the independent bivariate sources, and (ii) by subsequently decorrelating the real and imaginary parts of the bivariate independent components. To 'rectify' the second step (ii), observe that the pseudocovariance matrix of independent sources can be expressed as $\mathbf{P} = \mathbf{\Delta}_P\mathbf{\Lambda}_P$, to give

$$\begin{aligned} \mathbf{\Lambda}_P &= \boldsymbol{W}\mathbf{H}\mathbf{P}\mathbf{H}^T\boldsymbol{W}^T && (10)\\ &= \boldsymbol{W}\mathbf{H}(\mathbf{\Delta}_P^{1/2}\mathbf{\Lambda}_P\mathbf{\Delta}_P^{1/2})\mathbf{H}^T\boldsymbol{W}^T && (11) \end{aligned}$$

where $\mathbf{H}$ is the mixing matrix, $\boldsymbol{W}$ denotes the unmixing matrix obtained from SUT, and $\mathbf{\Delta}_P$ and $\mathbf{\Lambda}_P$ are matrices that convey the phase and magnitude information for each source as in (1). Subsequently,

$$\begin{aligned} \boldsymbol{W}\mathbf{H}\mathbf{\Delta}_P^{1/2} &= \mathbf{I}\\ \boldsymbol{W}\mathbf{H} &= \mathbf{\Delta}_P^{-1/2} && (12) \end{aligned}$$

$$\boldsymbol{W} = \mathbf{\Delta}_P^{-1/2}\mathbf{H}^{-1} \tag{13}$$

which implies that

$$\boldsymbol{W}^{-1} = \mathbf{H}\mathbf{\Delta}_P^{1/2} \tag{14}$$

Upon dividing each element of $\boldsymbol{W}^{-1}$ by its magnitude, the phase matrix $\mathbf{\Omega}_P$ can be obtained as

$$\mathbf{\Omega}_P = \begin{bmatrix} \pm\exp(\imath\theta_1) & \cdots & \pm\exp(\imath\theta_n)\\ \vdots & \ddots & \vdots\\ \pm\exp(\imath\theta_1) & \cdots & \pm\exp(\imath\theta_n) \end{bmatrix} \tag{15}$$

[1] Real-valued singular values mean that there is at least a phase ambiguity in the columns of $\mathbf{U}$ and $\mathbf{V}$.

[2] The pseudocovariance matrix is normalised by the covariance matrix as $\mathbf{C}^{-\frac{1}{2}}\mathbf{P}\mathbf{C}^{-\frac{1}{2}^T} = \mathbf{Q}_P\mathbf{\Lambda}_P\mathbf{Q}_P^T$; see [9], for more detail.

whose diagonal[3] matrix is $\text{diag}\{\mathbf{\Omega}_P\} = \mathbf{\Delta}_P^{1/2}$ in (14), that is,

$$\mathbf{\Delta}_P^{1/2} = \begin{bmatrix} \exp(\imath\theta_1) & & \phi \\ & \ddots & \\ \phi & & \exp(\imath\theta_n) \end{bmatrix}$$

To mitigate the ill-posedness of estimating the phase ambiguity [2][10], we here make the assumption that the mixing matrix is real-valued and the sources are complex-valued. This is realistic and is similar to using the sparsity assumption when solving under-determined source separation. Observe that, if $\mathbf{\Delta}_P$ is not of the form in (15), that is, when not all the elements of the kth column are $\pm\exp(\imath\theta_k)$ but assume the value of unity, then $\mathbf{\Delta}_P^{1/2} = \mathbf{I}$. For this particular case, CPT simplifies into SUT as shown later in (16).

Remark#4: Using (14), the $\pm$ sign ambiguity problem is completely bypassed, as we can reconstruct the bivariate sources irrespective of this ambiguity. Moreover, the scaling ambiguity is also bypassed by the normalisation of each element of $\boldsymbol{W}^{-1}$.

The proposed correlation preserving transform (CPT) therefore assumes the form

$$\boldsymbol{V} = \mathbf{\Delta}_P^{1/2}\mathbf{W} \tag{16}$$

where the strong uncorrelating transform $\mathbf{W}$ can be calculated from (9).

Remark#5: The computational cost of the CPT in (16) is similar to that of SUT, that is, $\mathcal{O}(N^3)$. This is because the cost of premultiplication of the SUT matrix $\mathbf{W}$ with the *diagonal* matrix $\mathbf{\Delta}_P^{1/2}$ is marginal compared to the computational complexity of calculating the two singular values in SUT.

The advantages of accounting for phase information by CPT are supported by the simulation studies in Section III.

III. Simulations

We next demonstrate the ability of the proposed Correlation Preserving Transform to resolve the intraference for signals with a nonvanishing phase angle, a typical case in practical applications. For a fair comparison, we benchmark the CPT against the performance of the strong uncorrelating transform.

For convenience, the same simulation setting as in [1] was adopted, where we controlled the degree of correlation, ρ, between the real and imaginary part of a complex-Gaussian zero mean signal $x = x_R + \imath x_I$, so as to generate noncircular signals spanning a range of a complex-valued pseudocovariances. To this end, two zero mean uncorrelated variables x_1 and x_2 were used to generate the real (x_R) and imaginary (x_I) part of the complex-valued signal as

$$x_R(t) = x_1(t), \quad x_I(t) = \rho x_1(t) + x_2(t)\sqrt{1-\rho^2} \tag{17}$$

Finally, the sources used in simulations were generated from

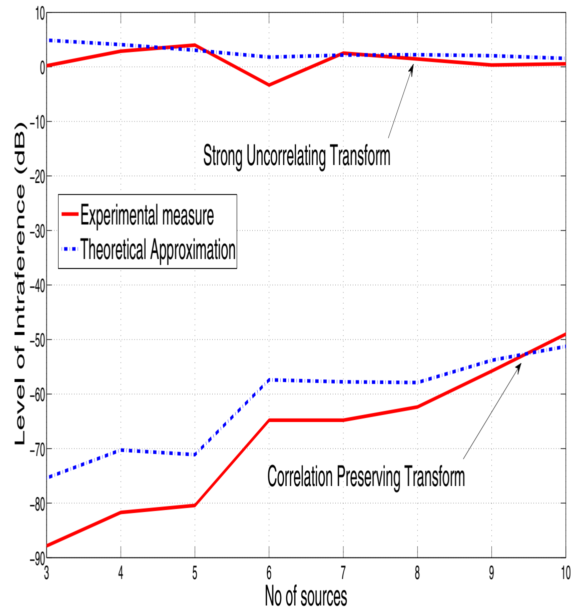

Fig. 1. Intraference as a function of the number of sources.

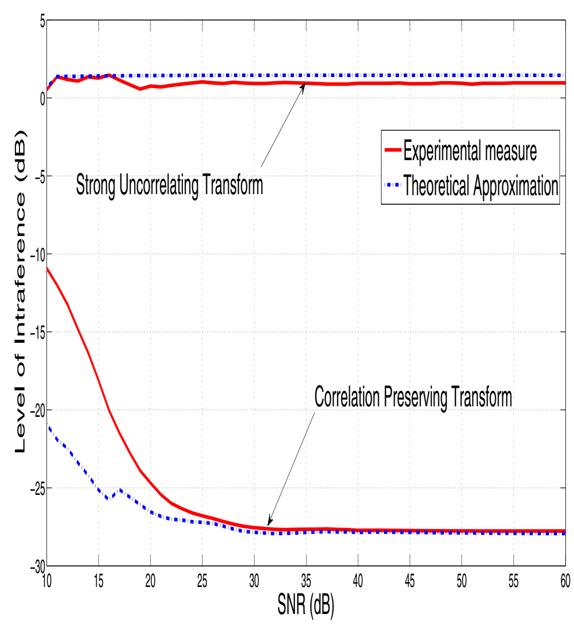

Fig. 2. Performance in terms of signal-to-noise ratio.

[3]$\mathbf{H}$ is post-multiplied by $\mathbf{\Delta}_P^{1/2}$ in (14).

a stable MA model

$$y(n) = x(n) + 0.8x(n-1) + 0.2x(n-2) \tag{18}$$

with real coefficients, as a complex filtering process would unnecessarily alter the statistics of the signal x. The parameters of the mixing matrix $\mathbf{H}$ were generated randomly from a centred Gaussian distribution.

A. Performance index: Intraference

The intraference for the kth bivariate source can be defined in terms of the deviation of the diagonal term $[k, k]$ of the mixing-unmixing model $\mathbf{VH}$ from unity, that is, based on

$$E\{\Phi^2[k,k]\} = E\{|(\mathbf{VH})[k,k] - 1|^2\} \tag{19}$$

This definition assumes without loss in generality that there is no permutation or scaling ambiguity. For simulation purposes, to circumvent the permutation ambiguity, sources were sorted in a descending order of their singular values (of the pseudocovariance matrix). As for the scaling ambiguity, although CPT can recover the sources irrespective of this ambiguity (see Remark 3), the definition of intraference in (19) assumes that $\mathbf{VH} = \mathbf{I}$. In our simulations, the rows of the unmixing matrix $\mathbf{V}$ were therefore scaled so that the diagonal elements of $\mathbf{VH}$ have unit norm.

The theoretical approximation of intraference in terms of the pseudocovariance can be evaluated using (2). Both measures of intraference (2) and (19) were evaluated and their mean value were measured, that is, $\frac{1}{N}\sum_{i=1}^{N}\Phi^2[i,i]$. The effects of intraference were studied comprehensively over two case studies: (a) performance for a varying number of sources, and (b) performance under additive noise. For case (a), Fig. 1 shows that the theoretical value of intraference (broken line) follows very closely the dynamics of the actual intraference. The small deviation is due to the small error assumption used in the analysis, which was based on a first order approximation. This also explains why for case (b) - when noise was prominent - there was also a difference between the theoretical approximation and the actual intraference.

B. Performance analysis for a varying number of sources

In the first set of simulations, the source separation problem was considered beyond the 2×2 case. Fig. 1 demonstrates the poor performance of SUT, highlighting the problems arising from not addressing the intraference. The excellent performance of CPT, on the other hand, demonstrates that we can recover the sources successfully if we cater for the modelling of intrinsic correlation within each complex-valued source. The theoretical approximation of the intraference followed quite closely the actual intraference measured in (19); the difference between the theoretical approximation and the actual intraference (in the case of CPT) decreased as the number of sources increased, indicating that CPT is a *consistent estimator*. The estimation of the phase angle for the kth source was calculated by taking the average of the kth column of the matrix in (15).

C. Performance analysis in the presence of noise

Fig. 2 illustrates the robustness of the proposed correlation preserving transform (CPT) in the presence of white Gaussian noise, over a range of signal to noise ratios (SNR). The performance advantage of CPT over SUT demonstrates that preserving the intrinsic correlation of a source can resolve the phase ambiguity in complex-valued independent component analysis, leading to a negligible level of intraference and thus preserving the integrity of the recovered sources. For SNR levels of less than 25 dB, the small error assumption holds [1] and, the theoretical approximation thus followed closely the actual intraference.

IV. Conclusions

We have proposed a solution to the problem of intraference in the context of complex-valued independent component analysis. The Correlation Preserving Transform (CPT) has been introduced to preserve the intrinsic correlation within the recovered complex-valued sources. In doing so, we have solved the well-known phase ambiguity problem, which is not trivial, as on one hand there should be a minimal level of correlation between the complex-valued independent components, and on the other hand, the correlation within each independent component had to be preserved to maintain the integrity of the sources. Simulations have demonstrated the efficacy of CPT, and have highlighted the shortcomings of current techniques that arise from not catering for the complex-valued nature of the pseudocovariance.

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