Real-Time Estimation of Quaternion Impropriety

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***Abstract*—An algorithm for tracking the degree of quaternion impropriety in real-time is proposed. This is achieved by exploiting the i-, j-, and k-pseudo-covariances, which makes it possible to introduce an impropriety measure as the minimum mean square error (MMSE) solution for estimating the quaternion involutions along the i, j, and k axes from the quaternion random variable itself. For rigour, convergence conditions for both the mean and mean square behavior of such a real-time impropriety tracker are established, and the effect of the degree of impropriety on the steady-state performance of the algorithm is analyzed. The concept is supported by simulations on both synthetic data and on real-world wind data recordings.**

***Index Terms*—Quaternion adaptive filtering, quaternion properness, quaternion impropriety measure, tracking of quaternion impropriety.**

I. Introduction

Quaternions have become a standard in a number of research areas, ranging from virtual reality to aeronautics [1]; furthermore, quaternions are gaining increasing popularity in engineering applications [2]-[6]. These applications have also revealed that the standard covariance, $c_{qq} = E[qq^*]$, of a quaternion random variable can only partially describe its second-order statistics and that estimation methods based on the standard covariance alone are optimal only for a special class of quaternion random variables, known as circular or proper, for which the probability distribution is rotation invariant [7]-[8].

Advances in quaternion statistics have established that in order to fully capture the second-order statistics of quaternion random variables, the standard covariance needs to be augmented with the three pseudo-covariances $c_{qq^i} = E[qq^{i*}]$, $c_{qq^j} = E[qq^{j*}]$, and $c_{qq^k} = E[qq^{k*}]$, referred to as the i-pseudo-covariance, j-pseudo-covariance, and k-pseudo-covariance. Furthermore, the augmented quaternion statistics and widely linear modeling of quaternion random variables (see [7]-[8]) have served as an enabling technology for algorithms that can cater for both second-order circular and non-circular signals, such as the widely linear quaternion least mean square (WL-QLMS) algorithm [9].

In contrast to complex random variables, properness of quaternion random variables has not yet been thoroughly addressed. Early attempts to define properness of quaternion random variables were based on the properties of probability distribution functions (pdf). The approach in [10] considers quaternion properness as the invariance of the pdf under specific rotations. A similar approach was taken in [11], where the condition for quaternion properness was the invariance of the pdf under rotations around any axis and for any angle.

Three different types of quaternion properness, based on vanishing of three different pseudo-covariances, and their implications on signal processing methods were analyzed in [8],[12] and an impropriety measure for each type of impropriety was proposed based on the Kullback-Leibler divergence between multivariate quaternion Gaussian distributions in [13]. A unified framework for second-order statistics of quaternion variables was presented in [7]-[8]; however, a real-time algorithm for tracking quaternion impropriety is still lacking.

Quaternion widely linear modeling is based on augmenting the quaternion random variable with its involutions along the i, j, and k axes [9]. Thus, quaternion widely linear algorithms have four times as many parameters as their strictly linear counterparts; when applied to quaternion proper signals, in order to achieve the same steady-state performance, the higher number of updates that have to be calculated in widely linear algorithms results in a higher computational cost. The larger number of parameters can also result in a higher gradient noise in gradient-based learning methods and slower convergence. Therefore, identifying the degree of impropriety of a signal is essential in both detection and estimation applications, to identify the instants where non-stationary signals change their statistics and to select an estimator that best suits the data.

To this end, we employ quaternion adaptive filtering to introduce a novel real-time quaternion impropriety tracking algorithm. Three quaternion impropriety measures based on the i-, j-, and k-pseudo-covariances are introduced and it is illustrated that each impropriety measure is the minimum mean square error (MMSE) solution for estimating the involutions of a quaternion random variable along the i, j, and k axes from the quaternion random variable itself. The mean and mean square behavior of the proposed algorithm are analyzed and the results are verified through simulations on both synthetic and real-world data.

II. Quaternion algebra and statistics

A quaternion variable $q \in \mathbb{H}$ comprises a real part $\Re(\cdot)$ and an imaginary part, or pure quaternion, $\Im(\cdot)$ consisting of three imaginary components, so that

$$q = \underbrace{q_r}_{\Re(q)} + \underbrace{iq_i + jq_j + kq_k}_{\Im(q)} \tag{1}$$

where $q_r, q_i, q_j, q_k \in \mathbb{R}$. The unit vectors i, j, k are also imaginary units and obey the following product rules

$$\begin{aligned} ij = k \quad jk = i \quad ki = j \\ i^2 = j^2 = k^2 = ijk = -1 \end{aligned} \tag{2}$$

making the multiplication non-commutative. The quaternion conjugate is given by $q^* = \Re(q) - \Im(q)$, and the norm by $|q| = \sqrt{qq^*} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}$.

The involution of $q \in \mathbb{H}$ around $\zeta \in \mathbb{H}$ is defined as $q^\zeta = \zeta q \zeta^{-1}$ [14]. For the case of i-, j-, and k-involutions, the four quaternion components can be expressed as [7]

$$\begin{aligned} q_r &= \frac{1}{4}\left(q + q^i + q^j + q^k\right) \quad q_i = \frac{1}{4i}\left(q + q^i - q^j - q^k\right) \\ q_j &= \frac{1}{4j}\left(q - q^i + q^j - q^k\right) \quad q_k = \frac{1}{4k}\left(q - q^i - q^j + q^k\right). \end{aligned} \tag{3}$$

With this definition of the ζ-involutions, for $\zeta \in \{i, j, k\}$, it can be proven that $q^{\zeta *} = q^{*\zeta}$ and $q\zeta = \zeta q^{\zeta^*}$.

In order to make the quaternion augmented statistics suitable for dealing with both proper and improper signals, a one-to-one relation is next established between the quaternion variable $q = q_r + iq_i + jq_j + kq_k$ and its real-valued vector representation $[q_r, q_i, q_j, q_k] \in \mathbb{R}^4$. From (3), the augmented quaternion variable $\mathbf{q}^a = [q, q^i, q^j, q^k]^T$ is related to its real-valued vector counterpart $\mathbf{q}^R = [q_r, q_i, q_j, q_k]^T$ as

$$\underbrace{\begin{bmatrix} q \\ q^i \\ q^j \\ q^k \end{bmatrix}}_{\mathbf{q}^a} = \underbrace{\begin{bmatrix} 1 & i & j & k \\ 1 & i & -j & -k \\ 1 & -i & j & -k \\ 1 & -i & -j & k \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} q_r \\ q_i \\ q_j \\ q_k \end{bmatrix}}_{\mathbf{q}^R} \tag{4}$$

where the matrix $\mathbf{A}$ is an invertible mapping between $\mathbf{q}^R$ and $\mathbf{q}^a$. The augmented covariance matrix now becomes

$$\mathbf{C}_q^a = E[\mathbf{q}^a \mathbf{q}^{aH}] = \begin{bmatrix} c_{qq} & c_{qq^i} & c_{qq^j} & c_{qq^k} \\ c_{q^i q} & c_{q^i q^i} & c_{q^i q^j} & c_{q^i q^k} \\ c_{q^j q} & c_{q^j q^i} & c_{q^j q^j} & c_{q^j q^k} \\ c_{q^k q} & c_{q^k q^i} & c_{q^k q^j} & c_{q^k q^k} \end{bmatrix} \tag{5}$$

where $\forall \zeta, \zeta' \in \{1, i, j, k\}, c_{q^{\zeta'} q^\zeta} = E[q^{\zeta'} q^{\zeta *}]$, and $c_{q^{\zeta'} q^\zeta} = c^*_{q^\zeta q^{\zeta'}}$. The following properties of the augmented covariance matrix in (5) will be used in this work.

Property 1: The diagonal elements in (5) are all different involutions of c_{qq}.

Proof: $c_{q^\zeta q^\zeta} = E[\zeta q \zeta \zeta q^* \zeta] = -\zeta E[qq^*]\zeta = c^\zeta_{qq}$. ■

Property 2: The cross-covariance between the involutions of q around two of the imaginary units ζ and ζ', is equal to the involution around the imaginary unit ζ of the cross-covariance between q and its involution around the third imaginary unit ζ'', that is $c_{q^j q^k} = c^j_{qq^i}$.

Proof: $c_{q^j q^k} = E[jqjkq^*k] = E[jqjijq^*ij] = -jE[qq^{i*}]j = c^j_{qq^i}$ (proofs for the other cases follow similarly). ■

Therefore, the complete second-order information within the augmented covariance matrix is contained in the standard covariance, c_{qq}, and the i-pseudo-covariance, c_{qq^i}, j-pseudo–covariance, c_{qq^j}, and k-pseudo-covariance, c_{qq^k}.

A. Quaternion impropriety measures

Properness of a quaternion random variable reflects the ratio of signal powers (and/or correlation) between the components of a quaternion random variable. Quaternion properness can be related to the properness of the projection of a quaternion random variable, q, on the six complex planes denoted by $\{1, i\}, \{1, j\}, \{1, k\}, \{i, j\}, \{i, k\}, \{j, k\}$ where '1' represents the real axis [7]-[8]. Thus, measuring the complex impropriety in these six planes measures the properness of the quaternion random variable. For complex random variables, the impropriety measure (known as the circularity quotient) is defined as the ratio between the pseudo-covariance and the covariance [15]. From Properties 1 and 2, notice that the six complex impropriety measures corresponding to the six pairs of axes can be extracted from the i-, j-, and k-pseudo-covariances. Thus, similar to the approaches in [7]-[8] and [12]-[13] we can now define the following three impropriety measures $\rho_\zeta \in \{\rho_i, \rho_j, \rho_k\}$ for quaternion random variables

$$\begin{aligned} \rho_i &= E[qq^*]^{-1} E[qq^{i*}] \\ \rho_j &= E[qq^*]^{-1} E[qq^{j*}] \\ \rho_k &= E[qq^*]^{-1} E[qq^{k*}] \end{aligned} \tag{6}$$

These measures reflect the correlation between q and each of its involutions, normalized by the signal power, $E[qq^*]$.

III. Relation between a quaternion variable and its involutions

Consider a linear mapping between a zero-mean quaternion random variable, q, and its involution, q^ζ, given by

$$\hat{q}^\zeta = h^* q \tag{7}$$

where $\zeta \in \{i, j, k\}$, $\hat{q}^\zeta$ is the estimate of q^ζ from q, and the coefficient h relates q to its ζ-involution. The goal is to find the optimal value for h, denoted by h_{opt}, that minimizes the mean square error (MSE) $E\left[|e|^2\right] = E\left[|\hat{q}^\zeta - q^\zeta|^2\right]$. Then, h_{opt} is the Wiener solution to (7), giving [7]

$$h_{opt} = E[qq^*]^{-1} E[qq^{\zeta *}] \tag{8}$$

Note the physical meaning of h_{opt} in (8) is the impropriety measure ρ_ζ given in (6). However, finding the Wiener solution requires knowledge of the true statistics of the data, which in general is not available. Moreover, block-based estimators for the Wiener solution are inadequate for non-stationary signals, which require an adaptive impropriety estimator.

A. Real-time impropriety estimation

Defining the impropriety measure as the optimal Wiener solution for estimating q^ζ from q allows for the use of strictly linear quaternion-LMS (QLMS) adaptive filters to track the impropriety measure in real time (here we use the iQLMS form [16]) . The algorithm for tracking ρ_ζ is therefore described by

$$\begin{aligned} \hat{q}^\zeta_n &= h^*_n q_n \\ e_n &= q^\zeta_n - \hat{q}^\zeta_n \\ h_{n+1} &= h_n + \tfrac{\mu}{2} q_n e^*_n \end{aligned} \tag{9}$$

where $\mu \in \mathbb{R}^+$ is the step size. As QLMS uses instantaneous estimates of the data statistics, the weights never reach their optimal values in the absolute sense, and it is important to analyze the contribution of the bias and variance, of the parameter estimates to the total MSE. To understand the behavior of the proposed impropriety tracker, we shall consider the weight error given by

$$\nu_n = h_n - h_{opt}. \tag{10}$$

B. Mean weight error behavior

To obtain the range for μ that ensures stable performance of the proposed impropriety tracker, from (9) we have

$$h_{n+1} - h_n = \frac{\mu}{2} q_n e_n^*. \tag{11}$$

Replacing (10) into the error $e_n^* = q_n^{\zeta *} - q_n^* h_n$ gives

$$e_n^* = q_n^{\zeta *} - q_n^* \nu_n + q_n^* h_{opt} \tag{12}$$

while substituting (12) into (11) yields

$$\nu_{n+1} - \nu_n = \frac{\mu}{2} q_n (q_n^{\zeta *} - q_n^* h_n)$$

$$\nu_{n+1} = \nu_n + \frac{\mu}{2} q_n q_n^{\zeta *} - \frac{\mu}{2} q_n q_n^* \nu_n - \frac{\mu}{2} q_n q_n^* h_{opt}. \tag{13}$$

Taking the statistical expectation of (13) and replacing h_{opt} with the expression in (8), we arrive at

$$E[\nu_{n+1}] = E[\nu_n]\Big(1 - \frac{\mu}{2} E[q_n q_n^*]\Big).$$

Therefore, the weight error converges to zero for $\big|1 - \frac{\mu}{2} E[q_n q_n^*]\big| < 1$, so that the allowable range for the step size becomes

$$0 < \mu < \frac{4}{E[q_n q_n^*]} = \frac{4}{\sigma_{q_n}^2}. \tag{14}$$

Remark 1: Note that if the condition in (14) is satisfied, the algorithm is asymptotically unbiased.
Remark 2: Convergence in the mean is not influenced by the degree of impropriety of the input signal.

C. Mean square weight error behavior

Consider the variance of the weight error in the steady-state, given by

$$E\big[v_{n+1} v_{n+1}^*\big] = E\big[|\nu_{n+1}|^2\big] = E\big[|\nu_n + \frac{\mu}{2} q_n e_n^*|^2\big].$$

Replacing $e_n^* = q_n^{\zeta *} - q_n^* h_n = q^{\zeta *} - q^*(\nu_n + h_{opt})$ into the equation above gives

$$E[|\nu_{n+1}|^2] = E\Big[\Big|\nu_n + \frac{\mu}{2} q_n \big(q_n^{\zeta *} - q_n^*(v_n + h_{opt})\big)\Big|^2\Big].$$

For an algorithm that converges in the mean, in the steady-state $h_{n+1} \simeq h_n$, so that $E[\nu_n] \simeq 0$, which gives

$$E\big[|\nu_{n+1}|^2\big] = E\big[|\nu_n|^2\big]\big(1 + \frac{\mu^2}{4} E\big[|q_n q_n^*|^2\big] - \mu E[q_n q_n^*]\big) + \frac{\mu^2}{4}\xi \tag{15}$$

where

$$\xi = E\big[|q_n q_n^{\zeta *}|^2\big] + E\big[|q_n q_n^*|^2\big] E\big[|h_{opt}|^2\big] - 2E\left[\Re(q_n q_n^{\zeta *} h_{opt}^* q_n^* q_n)\right]. \tag{16}$$

In the steady-state, $E\big[|\nu_{n+1}|^2\big] \simeq E\big[|\nu_n|^2\big]$, and the expression in (15) simplifies into

$$E\big[|\nu_n|^2\big] = \frac{\frac{\mu}{4}\xi}{E[q_n q_n^*] - \frac{\mu}{4} E\big[|q_n q_n^*|^2\big]}. \tag{17}$$

For convergence in the mean square, the steady-state weight error variance, $E\big[\,|v_n|^2\,\big]$, given in (17) must remain positive and bounded. This is satisfied for $E[q_n q_n^*] - \frac{\mu}{4} E\big[|q_n q_n^*|^2\big] > 0$ which gives the condition

$$0 < \mu < 4 \frac{E[q_n q_n^*]}{E\big[|q_n q_n^*|^2\big]}$$

where expressing the term $E\big[|q_n q_n^*|^2\big]$ in terms of the second-order statistics gives

$$0 < \mu < \frac{8}{\sigma_{q_n}^2 \big(3 + |\rho_i|^2 + |\rho_j|^2 + |\rho_k|^2\big)}. \tag{18}$$

Remark 3: From (17) and (18), observe that the mean square behavior of the proposed quaternion impropriety tracker depends on the degree of impropriety, as the terms ξ and $E\big[|q_n q_n^*|^2\big]$ contain impropriety information.

IV. Simulations

A. Synthetically generated data

The impropriety tracking ability of the algorithm is first demonstrated on a synthetically generated signal constructed from three segments of zero-mean white Gaussian noises with varying pseudo-covariances (and hence impropriety measures) and unit power. Figure 1 shows the absolute values of the pseudo-covariances for the data segments with different improprieties, together with their estimates, for $\mu = 10^{-1}$. Observe that the algorithm produced accurate impropriety estimations for various types of impropriety.

To illustrate the ability of the proposed impropriety tracker to identify impropriety in one dimension, we considered j-impropriety where ρ_j was set to 0.65 for the first segment, 1 for the second segment, and 0.3 for the third segment. Figure 2 shows 100 realizations of the estimate of ρ_j and their average, demonstrating that the proposed algorithm produces unbiased estimates and that mean convergence is not affected by the degree of impropriety; this verifies Remarks 1 and 2. Moreover, observe that the steady-state variance of the impropriety tracker is dependent on the degree of impropriety (see Remark 3).

B. Wind data

The wind speed measured in the north, east, and vertical directions comprised the pure quaternion part and the ambient temperature was used as the real part of the quaternion-valued wind signal [6]. The recorded wind signal exhibits high degree of non-circularity, as seen in the scatter diagram in Figure 3. Figure 4 shows the impropriety measures corresponding to the wind regimes, with the step size set to $\mu = 10^{-1}$.

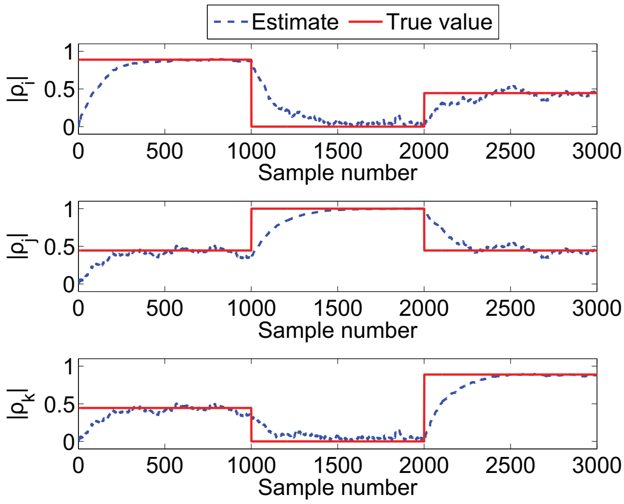

Fig. 1. Absolute impropriety measures for synthetically generated Gaussian data (red) and the impropriety estimates (blue).

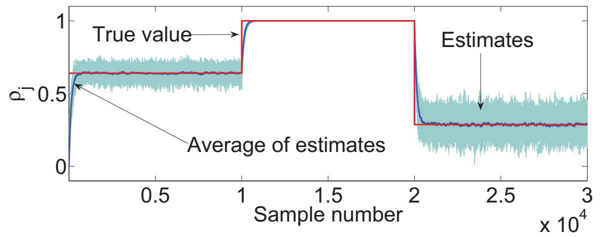

Fig. 2. True value of ρ_j (in red) plotted alongside 100 realizations of its estimate (in light green) and the average of the estimates (in blue).

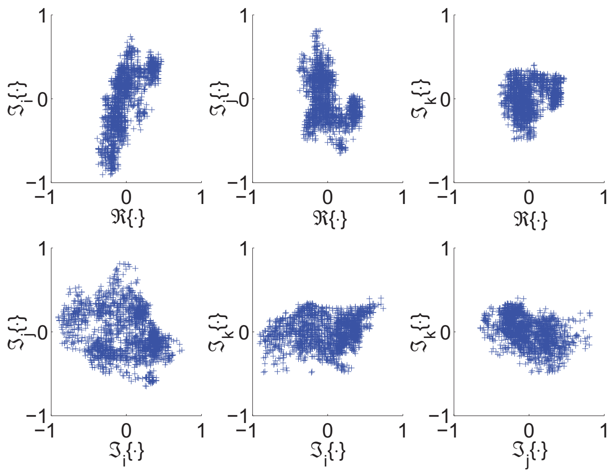

Fig. 3. Scatter diagram of the improper distribution of wind data.

C. Communication channel estimation

Finally, we considered a Multiple-Input-Multiple-Output (MIMO) wireless communication system based on Alamouti coding [17], whereby the coding scheme is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{19}$$

while y_1 and y_2 are two consecutive complex valued received signals, s_1 and s_2 are two consecutive complex valued transmitted signals, h_1 and h_2 are complex valued channel gains between each transmit antenna and the receiver, whereas η_1 and η_2 represent complex valued noise terms.

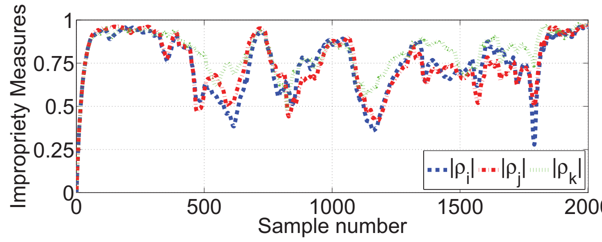

Fig. 4. Absolute value of the estimated impropriety measures of quaternion-valued wind data.

Using the Cayley-Dickson representation, two complex numbers can be combined into a quaternion, giving the quaternion form of the Alamouti code as $Y = HS + \mathcal{V}$ [18]-[19], where $Y = y_1 + jy_2$, $H = h_1 + jh_2$, $S = s_1 + js_2$, and $\mathcal{V} = \eta_1 + j\eta_2$. The circularity of the channel can be used to analyze its diversity and to establish whether any phase information can be extracted from the received signal [20]. For the first segment (0 to 2.5 seconds) the two complex valued channels in (19) were independent, one was proper complex and the other was improper complex with a complex impropriety measure of 8×10^{-1}. For the second segment (2.5 to 5 seconds) both channels were complex circular and had a cross-correlation of 4×10^{-1}. Figure 5 illustrates the ability of the proposed impropriety tracker to successfully track the changes in channel statistics; the step size was $\mu = 10^{-2}$ and the channel was measured every 5 milliseconds.

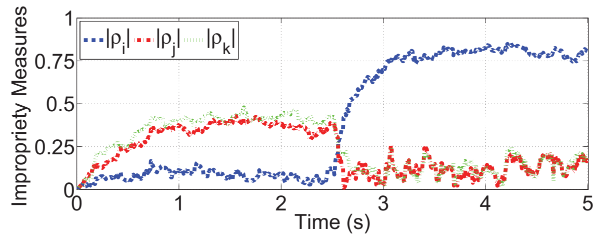

Fig. 5. Channel impropriety measure estimation of an Alamouti communication system.

V. Conclusion

A real-time tracker of quaternion impropriety has been introduced. This has been achieved based on the MMSE linear estimation of the involutions of a quaternion random variable along the i, j, and k axes from the quaternion random variable itself. Convergence conditions in the mean and mean square sense have also been obtained. The analysis has shown that the proposed algorithm produces unbiased estimates and that the mean behavior of the algorithm is not affected by the degree of impropriety. On the other hand, the steady-state variance of the algorithm has been shown to exhibit strong dependence on the degree of impropriety. The analysis has been verified using simulations on both synthetic and real-world data.

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