Selective Time-Frequency Reassignment Based on Synchrosqueezing

Alireza Ahrabian and Danilo P. Mandic

***Abstract*—Reassignment methods seek to sharpen the time-frequency representation of conventional time-frequency algorithms, such as the continuous wavelet transform (CWT). However, such methods aim to localize both noise components and signal components of interest, which makes the discrimination between such components for low SNR signals a difficult task. Inspired by the recovery of modes (RCM) algorithm, we propose a selective time-frequency reassignment procedure that attempts to identify and localize oscillatory components of interest for the continuous wavelet transform (CWT), where the reassignment is carried out for selective localization. The performance of the proposed method is illustrated on both synthetic and real world data.**

***Index Terms*—Continuous wavelet transform, reassignment methods, synchrosqueezing transform, time-frequency analysis.**

I. Introduction

THE analysis of nonstationary signals has traditionally been carried out using time-frequency methods, such as the short-time Fourier and continuous wavelet transforms. However, due to the uncertainty principle [1], such techniques are fundamentally limited in simultaneously resolving oscillatory components in both time and frequency. A number of approaches have been proposed in order to overcome this limitation, from data driven methods such as the empirical mode decomposition (EMD) [2] and its multivariate extensions [3][4] to reassignment methods such as the synchrosqueezing transform (SST) [5].

The empirical mode decomposition decomposes a signal into a set of amplitude and frequency modulated monocomponent signals termed intrinsic mode functions (IMFs). Owing to the narrowband properties of IMFs, the Hilbert transform can then be applied in order to obtain physically meaningful instantaneous amplitudes and frequencies so as to construct a time-frequency representation. The empirical mode decomposition has found many applications as both a time-frequency algorithm and in signal decomposition in fields such as biomedical signal processing [6]. It should be noted that, while data driven methods have shown great promise in overcoming the limitations of conventional time-frequency methods, the non-parametric nature of such methods does not admit closed form analysis of their convergence and stability.

Reassignment techniques [7]–[11] such as the synchrosqueezing transform (SST) [5] have been proposed as an alternative to data driven methods. The synchrosqueezing was introduced as a post-processing technique applied to the continuous wavelet and short time Fourier transforms [12] in order to better localize oscillatory components. This was achieved by inverting the linear projection based transforms (DTF, wavelet) around the common instantaneous frequency estimates in each coefficient of such transforms. The synchrosqueezing transform enhances the time-frequency representation of signals in applications such as condition monitoring [13], however, for mono/multicomponent signals with low signal to noise ratios, the synchrosqueezing transform localizes both the noise and desired signals, leading to time-frequency representations that are difficult to interpret. Furthermore, the computational resources required by the SST for signals of long duration is also a challenge which can, in many instances, be reduced by applying reassignment to a subset of the wavelet coefficients.

In order to extract oscillatory components from the time-frequency domain, conventional approaches first identify ridges, that is, the local maxima of the time-frequency coefficients with respect to scale or frequency. Ridge detection methods for both wavelet and synchrosqueezing transforms have been presented in [14][15], whereby a cost function is employed which both identifies the local maxima and smooths ridges corresponding to the instantaneous frequency estimates of oscillatory components of interest. However, such methods are computationally expensive and require greedy algorithms for the identification of such ridges. The work in [16] introduced a simultaneous mode extraction and denoising algorithm inspired by synchrosqueezing (referred to as the retrieval of components (RCM) algorithm) that outperformed both the empirical mode decomposition and the block thresholding wavelet transform in recovering modulated oscillations in noise. The method did not perform well for very low SNR conditions. More recently, the work in [17] has introduced a scheme for partitioning the the time-frequency domain for the STFT using techniques inspired by computational geometry in order to either reassign and/or extract the oscillations of interest.

This work introduces a selective reassignment method (the first of such methods was introduced in [18]), in order to reassign oscillatory components of interest while suppressing the noise components, using a variant of the retrieval of components (RCM) algorithm proposed in [16]. The proposed method

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The authors are with the Department of Electrical and Electronic Engineering, Imperial College London SW7 2AZ, U.K. (e-mail: alireza.ahrabian06@imperial.ac.uk; aa1106@ic.ac.uk; d.mandic@imperial.ac.uk).

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is shown to be computationally efficient, and the resulting improvement in the time-frequency representation is illustrated both on synthetic and real world data.

II. Problem Statement

Consider the following measurement model

$$x(t) = s(t) + n(t), \tag{1}$$

where $s(t)$ is the signal of interest and $n(t)$ the additive noise process. The continuous wavelet transform of $x(t)$ is then given by, $W_x(a,t) = W_s(a,t) + W_n(a,t)$, where $W_s(a,t)$ is the continuous wavelet transform of the desired signal, while $W_n(a,t)$ are the CWT coefficients for the noise process. However, conventional reassignment methods, such as the synchrosqueezing transform [5], localize the wavelet coefficients $W_x(a,t)$ which contain both the noise process and desired signal. This approach has two drawbacks: (i) for very low SNR signals, visualising the oscillatory components is more difficult; (ii) the computational resources used by synchrosqueezing, when analyzing signals of long duration, are rather prohibitive. Our aim is to show that by identifying a subset of the wavelet coefficients to reassign, a significant reduction in the computational and memory requirements can be obtained by eliminating the localization of noise components.

The objective of the proposed selective reassignment is therefore to both identity and localize the wavelet coefficients corresponding to the desired signal, $W_s(a,t)$.

III. Retrieval of Components from a Multicomponent Signal (RCM) Algorithm

We next provide a brief introduction of the so-called retrieval of components from a multicomponent signal (RCM) algorithm [16]. The objective of the RCM algorithm is to identify oscillatory modes; the algorithm first identifies the number of oscillatory modes then an optimal threshold is determined such that oscillatory modes can be retrieved.

Within the RCM algorithm, the number of oscillatory modes, $C(\gamma,t)$, for various threshold levels, γ, and time instants t, is calculated as the following set[1]

$$C(\gamma,t) = \{j, |W(a_j,t)| < \gamma \quad \text{and} \quad |W(a_{j+1},t)| > \gamma\}, \tag{2}$$

where $\gamma \in \Gamma(t)$ is the set of thresholds which is bounded between the maximum value of the wavelet coefficients and $\sigma_n/2$ where σ_n is the standard deviation of the noise process estimated using the median absolute deviation of the finest wavelet coefficients[2] (an example is shown in Fig. 1). For each time instant, t, the number of modes is determined as

$$M(t) = \text{Mode}(V(t)), \tag{3}$$

where $V(t) = \{\#C(\gamma,t)\}, \forall\gamma$ and $\#$ is the operator which determines the number of elements in a set, while the function,

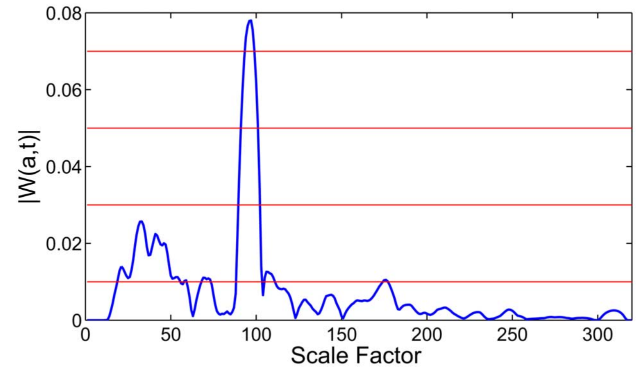

Fig. 1. The magnitude of the CWT coefficients (with various threshold levels (red thin line)) along the scale factors (blue thick line), for a fixed time instant.

$\text{Mode}(\cdot)$, is the statistical mode of a series of data points. The total number of oscillatory modes, N_f, is then determined as

$$N_f = \text{Mode}(N_s), \tag{4}$$

where $N_s = \{M(t)\}, \forall t$. Once the number of oscillatory modes has been obtained, the second step is to calculate a threshold such that the oscillatory modes can be retrieved. To this end, the work in [16] defines the following set

$$S(t) = \{\gamma \in \Gamma(t), \text{s.t.} \#C(\gamma,t) = N_f\}, \tag{5}$$

such that $S(t)$ is the set of thresholds, γ, in the set $\Gamma(t)$ at each time instant t, that yields the number of modes N_f. Finally, the optimal threshold for each time instant, $\hat{\gamma}(t)$, is given by $\hat{\gamma}(t) = \text{Median}(S(t))$. The oscillatory modes are then extracted by taking the inverse of the wavelet transform (a detailed explanation can be found in [16]).

IV. Proposed Algorithm

We propose a variation of the RCM algorithm which robustly identifies oscillatory modes pertaining to the signals of interest in high levels of noise such that the resulting identified oscillatory modes are reassigned using synchrosqueezing for selective localization. The proposed algorithm follows the procedure in [16] in terms of identifying the number of oscillatory modes, however, for each algorithmic step a modification is introduced.

In order to find the number of oscillatory modes, the wavelet coefficients, $W_x(a,t)$, are first partitioned along time into N_w equal non-overlapping windows of length W (where appropriate zero-padding of $W_x(a,t)$ is carried out if the original signal is not divisible by W). The absolute values of the wavelet coefficients in each non-overlapping window are then averaged across time, to give

$$\hat{W}_x(a,b) = \frac{1}{W}\sum_{t\in\Psi(b)} |W_x(a,t)|, \quad b = 1,\ldots,N_w, \tag{6}$$

where $\Psi(b)$ corresponds to the set of time indices within each window and the integer b denotes the index of each window. By averaging the CWT coefficients along time, the signals of interest tend to be more localized as compared with the noise components (see the Appendix). The next step is to identify the number of oscillatory modes within each window by using (2) and (4), where the coefficients $\hat{W}_x(a,b)$ are used instead of the

[1]Where $a_j = 2^{j/n_v}\Delta t$ for $j = 0,\ldots,Ln_v$ are the discrete scales. The term n_v is an input parameter, while $T = 2^{L+1}$ where T is the length of the signal with appropriate zero padding for L to be an integer.

[2]The proposed method uses a lower bound determined by the magnitude of the finest wavelet coefficient.

coefficients $W_x(a,t)$. The set of thresholds which obtain the same number of oscillatory modes (shown in (5)), is modified based on

$$S_m(b) = \{\gamma \in \Gamma(b), \text{ s.t. } \#C(\gamma, b) = M(b)\}. \quad (7)$$

This restriction allows for the processing of intermittent oscillations as well as for the reduction in computational resources, as selective reassignment is less restrictive in terms of finding globally the number of oscillatory modes. The optimal threshold, $\hat{\gamma}_m(b)$, is then determined as $\hat{\gamma}_m(b) = \mathrm{E}\{S_m(b)\}$, where $\mathrm{E}\{\cdot\}$ is the statistical expectation operator. In order to threshold the CWT coefficients, the following hard thresholding is applied

$$\begin{cases} \hat{W}_s(a,t) = W_x(a,t), & t \in \Psi(b) \quad \text{if} |\hat{W}_x(a,b)| \geq \hat{\gamma}_m(b), \\ \hat{W}_s(a,t) = 0, & t \in \Psi(b) \quad \text{if} |\hat{W}_x(a,b)| < \hat{\gamma}_m(b), \end{cases}$$

for all (a,b) pairs, and the final estimates of CWT coefficients corresponding to the signal of interest are given by $\hat{W}_s(a,t)$. Synchrosqueezing is then carried out to generate a selectively reassigned coefficients $\hat{T}(\omega_l,t)$. Owing to the hard thresholding of the wavelet coefficients prior to synchrosqueezing, the original signal can then approximately be recovered as $x_r(b) \approx 2R_\psi^{-1}\sum_l \Re[\hat{\mathfrak{T}}(\omega_\mathfrak{l},\mathfrak{b})]$, where R_ψ^{-1} is the normalization constant [15]. An alternative thresholding method for recovering the wavelet coefficients proposed in [19][16] first identifies the wavelet ridges greater than the thresholds $\hat{\gamma}_m(b)$. The wavelet coefficients around the vicinity of the ridge (determined by the compact support of the mother wavelet) are then used to recover the oscillatory modes.

V. Simulation Results

The time-frequency representations of selective reassignment were compared quantitatively and qualitatively with that of the standard synchrosqueezing transform[3], for both synthetic and real world data.

A. Synthetic Signals

The first set of simulations consists of a multicomponent frequency modulated oscillation, given by

$$y(t) = \cos 2\pi(20t + 6\cos t) + \cos 2\pi(35t + 0.663t^2) + n(t)$$

where the sampling frequency was 200 Hz and the signal duration was 10 seconds. The symbol $n(t)$ denotes a fractional Gaussian noise (fGn) process[4], with Hurst exponent $H = 0.8$ (where the energy of the noise process is concentrated at the lower frequencies). In order to quantitatively compare the performance of the proposed method with that of SST, we calculated, for various SNRs, the corresponding energy of the transforms both along the desired instantaneous frequency and for the whole time-frequency domain (not including the energy along the desired instantaneous frequency).

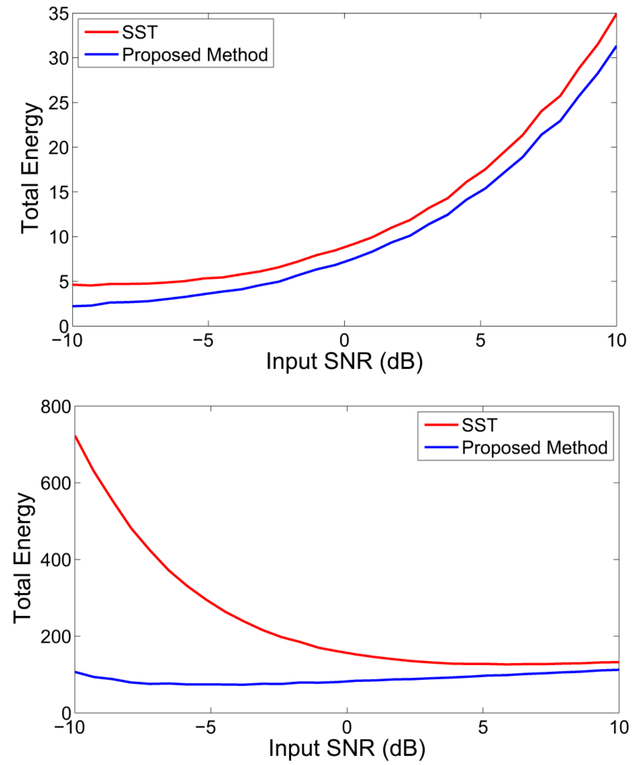

Fig. 2. Comparison of the proposed method and SST.(Upper panel) The total energy along the instantaneous frequency of interest.(Lower panel) The total energy across the time-frequency domain not including the instantaneous frequency of interest.

Fig. 2 (upper panel) shows the total energy along the desired instantaneous frequency for both the proposed method and SST. The energy for both methods decreased for very low SNR signals, however, the proposed method estimates a subset of the total energy estimated using the synchrosqueezing transform. This is due to only the subset of wavelet coefficients being reassigned; wavelet coefficients above a threshold are reassigned irrespective of the instantaneous frequency. Fig. 2 (lower panel) shows the total energy across the whole time-frequency domain while not including the energy along the desired instantaneous frequency, for various input SNRs. It can be observed that the proposed method significantly reduces the total background noise for lower SNRs. Fig. 3 illustrates the time-frequency representation for both the synchrosqueezing transform (upper panel) and the proposed method (lower panel), for the multicomponent signal with an SNR of 0 dB; the proposed method clearly recovers and accurately represents the oscillatory components of interest. Mode recovery algorithms [15] can also be used to retrieve smoothed estimates of the instantaneous frequency with lower computation costs.

B. Real World Data

This simulation considered steady state visual evoked potentials (SSVEP) electroencephalography (EEG) data collected from a test subject, where a 15 Hz blinking stimulus was used in the experiment. Data was recorded from the POz electrode, and bandpass filtering was then carried out between 2-50 Hz.

[3]The bump wavelet [16] was used with $\mu = 5$ and $\sigma = 1$. Furthermore, the support of the bump mother wavelet is dependent upon σ, where for small values of σ the frequency localization is enhanced at the expense of time localization, therefore it is imperative to select an appropriate σ.

[4]The Hurst exponent is bounded as, $0 < H < 1$, where $H = 0.5$ denotes a white Gaussian noise process. Between $0.5 < H < 1$ there exists positive correlation for different time lags, while for $0 < H < 0.5$ there is negative correlation for different time lags [20].

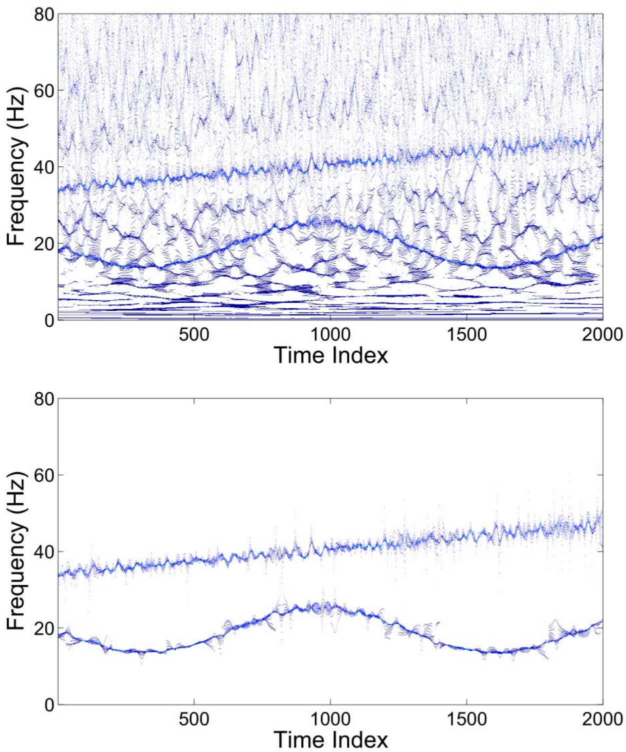

Fig. 3. Time-frequency representation of the multicomponent signal, $y(t)$, with SNR $=0$ dB, using SST (upper panel) and using the proposed method (lower panel).

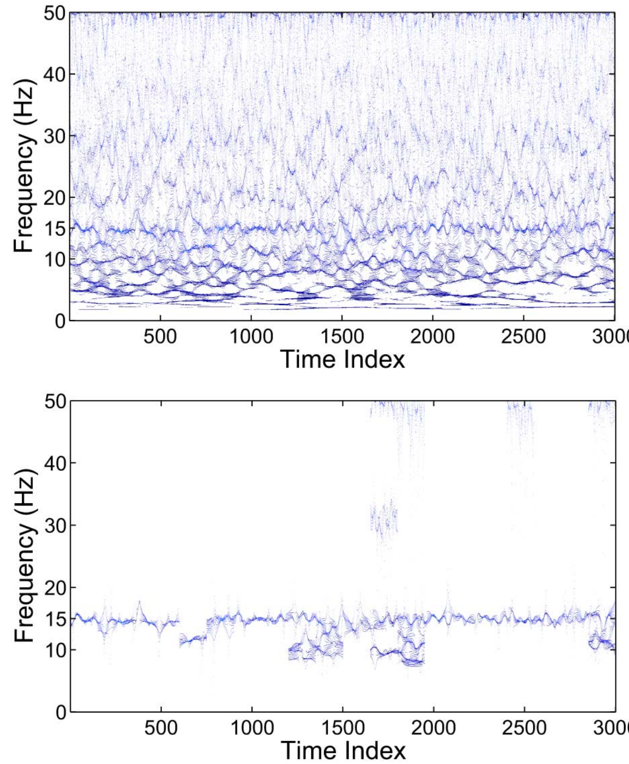

Fig. 4. Time-frequency representation of SSVEP using SST (upper panel) and the proposed method (lower panel).

The synchrosqueezing transform of the SSVEP data is shown in Fig. 4 (upper panel) where, due to the low SNR, the frequency signature of interest is not clearly represented. However, for the proposed method (shown in Fig. 4 (lower panel)), the 15 Hz SSVEP signal was clearly identified in the time-frequency domain.

VI. Conclusion

A selective reassignment procedure based on the RCM algorithm has been proposed to identify oscillatory components of interest from noisy data. The principles of synchrosqueezing have then been applied in order to generate a selectively localized time-frequency representation. The performance of the proposed method has been demonstrated on both synthetic and real world data.

Appendix

Consider a signal $x(t)=s(t)+n(t)$, such that, $s(t)=Ae^{i\phi(t)}$ and $n(t)=n_r(t)+in_i(t)$, where $n_r(t)$ and $n_i(t)$ are independent zero mean Gaussian noise processes [1], with a variance of σ^2. The magnitude of the averaged wavelet coefficients (shown in (6)) can be approximated as follows

$$\mathrm{E}\{|W_x(a,t)|\} \approx \mathrm{E}\left\{\left|a^{-1}\int_{-\infty}^{\infty} s(\tau)\psi\left(\frac{\tau-t}{a}\right)d\tau\right|\right\} +\mathrm{E}\left\{\left|a^{-1}\int_{-\infty}^{\infty} n(\tau)\psi\left(\frac{\tau-t}{a}\right)d\tau\right|\right\}. \quad (8)$$

From [21], an approximate expression for the wavelet transform of the signal $s(t)$ was obtained for a general multivariate modulated oscillation (a similar result is also valid for univariate modulated oscillations); furthermore, the wavelet transform of the noise process can also be simplified as follows

$$\mathrm{E}\{|W_x(a,t)|\} \approx \mathrm{E}\left\{\frac{A}{2}|\hat{\Psi}(a\phi'(t))|\right\} +a^{-1}\int_{-\infty}^{\infty}\mathrm{E}\{|n(\tau)|\}\left|\psi\left(\frac{\tau-t}{a}\right)\right|d\tau, \quad (9)$$

where $\hat{\Psi}(\cdot)$ is the Fourier transform of the mother wavelet and the absolute value of the noise signal follows a Rayleigh distribution, such that, $\mathrm{E}\{|n(\tau)|\}=\sigma\sqrt{\frac{\pi}{2}}$, to give

$$\mathrm{E}\{|W_x(a,t)|\} \approx \mathrm{E}\left\{\frac{A}{2}|\hat{\Psi}(a\phi'(t))|\right\} + a^{-1}\sigma\sqrt{\frac{\pi}{2}}\Psi_n(a), \quad (10)$$

where, $\Psi_n(a)=\int_{-\infty}^{\infty}|\psi(\frac{\tau-t}{a})|d\tau$. By averaging the absolute value of the CWT coefficients along time, (10) shows that the wavelet transform of the noise process (for sufficiently large number of samples averaged along time) results in wavelet coefficient values that are dependent upon both the standard deviation of noise process and the scaled mother wavelet. The expected value of the signal of interest is dependent primarily on the instantaneous frequency. For modulated oscillations containing slowly varying instantaneous frequencies, the expected value of the absolute value of the wavelet coefficients would be more localized over fewer scales and therefore more localized relative to the noise signal.

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