the Modulated E-spline with Multiple subbands and its application to sampling wavelet-sparse signals

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May 7, 2014

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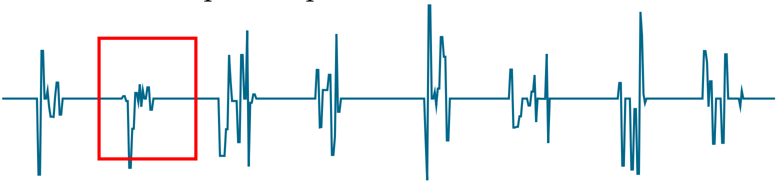
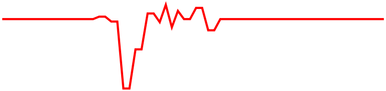
Motivation

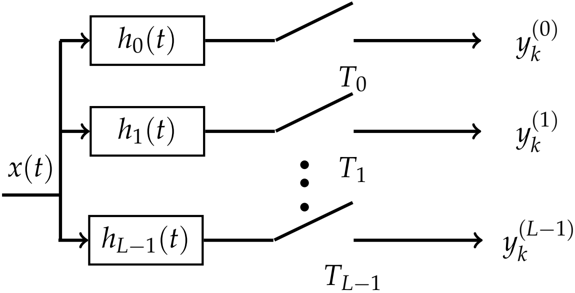
To have a realistic and physically realizable acquisition scheme for sub-Nyquist sampling sequences of pulses with unknown shapes. We assume that the pulses have compact support in time and have sparse representation in the wavelet domain.





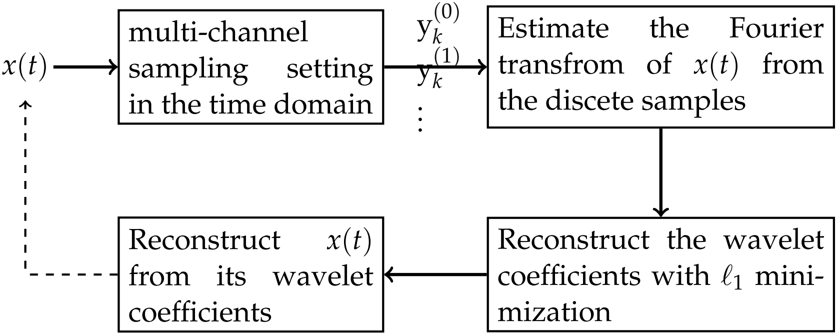
At the end, recover the pulse from its discrete samples

SUMMARY OF OUR SOLUTION



- $h_l(t)$ is the Modulated E-spline with Multiple subbands (MEMS), which is constructed by a new method
- MEMS allows stable estimation of the exact Fourier transform of the signal over a wide range of predefined frequencies
- ℓ_1 minimization for reconstruction

DIAGRAM OF OUR SOLUTION



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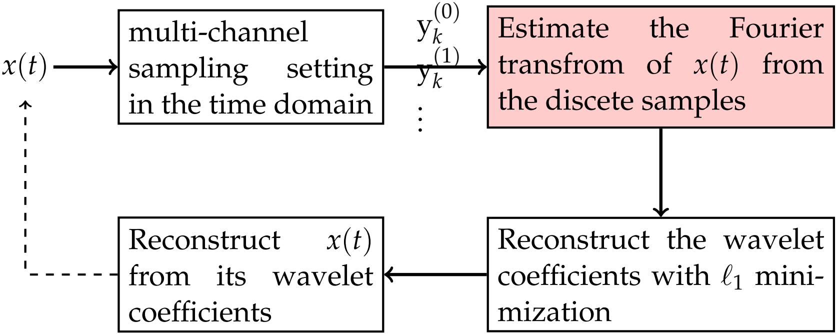
Roadmap – where we are



The streamline of this section

1. To estimate the Fourier transform from the discrete spatial samples – we need exponential reproducing kernels
2. E-spline – the exponential reproducing kernel central to the exponential reproducing concept
3. E-spline – not stable in the concerned scenario
4. Our solution – a new reconstruction method that gives us MEMS

1, 2 and 3 explain why we need MEMS.

Estimation of the Fourier transform

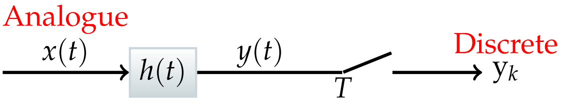


Figure: The sampling structure of one channel, $h(t) = \varphi(-t/T)$.

- The discrete samples at the output of one channel are

$$y_k = \langle x(t), \varphi(t/T - k) \rangle. \tag{1}$$

- The Fourier transform of $x(t)$ at predefined frequencies can be estimated from y_k if $\varphi(t)$ is a exponential reproducing kernel.

Exponential reproducing kernels

- An exponential reproducing kernel satisfies

$$\sum_{k\in\mathbb{Z}} c_k \psi(t-k) = e^{\alpha t} \text{ with } \alpha \in \mathbb{C}, \tag{2}$$

  i.e. it satisfies the generalized Strang-fix conditions (explained later).
- The convenient relationship between the discrete spatial samples and the Fourier transforms of the analogue signal. If $\alpha = j\omega$ for some real ω

$$s = \sum_{k\in\mathbb{Z}} c_k y_k = \sum_{k\in\mathbb{Z}} c_k \langle x(t), \psi(t/T-k)\rangle$$

moment

$$= \langle x(t), \sum_{k\in\mathbb{Z}} c_k \psi(t/T-k)\rangle = \boxed{\langle x(t), e^{j\frac{w}{T}}\rangle}. \tag{3}$$

Fourier transform

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E-SPLINES

- Exponential splines (E-Splines) are central to the exponential reproduction property.
- The E-spline is any function $\beta_{\boldsymbol{\alpha}}(t)$ with Fourier transform

Order of the E-spline

$$\hat{\beta}_{\boldsymbol{\alpha}}(\omega) = \prod_{n=0}^{P} \frac{1 - e^{\alpha_n - j\omega}}{j\omega - \alpha_n}. \tag{4}$$

- It can reproduce $e^{\alpha_n t}$, i.e. there exist weights $c_{n,k}$ such that

$$\sum_{k \in \mathbb{Z}} c_{n,k} \beta_{\boldsymbol{\alpha}}(t-k) = e^{\alpha_n t} \tag{5}$$

- $\gamma(t) * \beta_{\boldsymbol{\alpha}}(t)$ can also reproduce $e^{\alpha_n t}$.

The Strang-fix conditions

- When $\alpha_n = j\omega_n$ is purely imaginary, the Strang-fix conditions are simply:

$$\hat{\beta}_{\boldsymbol{\alpha}}(w_n) \neq 0 \tag{6}$$

$$\hat{\beta}_{\boldsymbol{\alpha}}(w_n + 2\pi l) = 0 \ (\forall l \neq 0), \tag{7}$$

Sufficient and necessary condition for $\beta_{\boldsymbol{\alpha}}(t)$ to be able to reproduce $e^{j\omega_n t}$

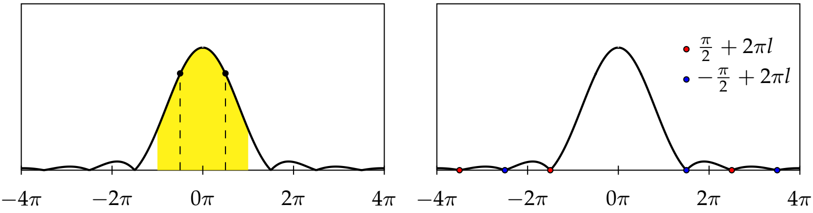
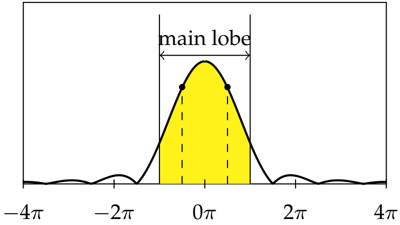


Figure: Illustration for the Strang-fix conditions. The amplitude of $\hat{\beta}_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = \pm j[\frac{\pi}{2}]$.

Observations



- main lobe size = 2π
- the generating parameters in ω have to stay in the main lobe
- real E-spline with generating parameters over π has instability issues

Why–Instability problems

Let $\boldsymbol{\omega}^+ = \{w : w \in \boldsymbol{\omega}, w > 0\}$, $\boldsymbol{\omega}^- = \boldsymbol{\omega} \setminus \boldsymbol{\omega}^+$,

$$\hat{\beta}_{\boldsymbol{\omega}}(w) = \boxed{\hat{\beta}_{\boldsymbol{\omega}^+}(w)}\,\boxed{\hat{\beta}_{\boldsymbol{\omega}^-}(w).} \tag{8}$$

multiplication of two radial functions with fast decaying tails - instability issue

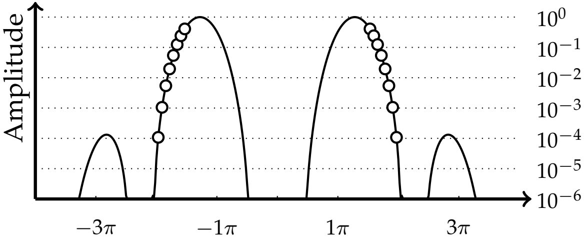


Figure: The conventional E-spline $\beta_{\boldsymbol{\omega}}(t)$ reproduce exponential e^{jwt} with $w \in \boldsymbol{\omega} = \{\frac{\pm k}{32} | k = 49 : 2 : 63\}$. $\beta_{\boldsymbol{\omega}}(t)$ is unstable because the ratio $\frac{\min_{w_n} |\hat{\beta}_\omega(w_n)|}{\max_{w_n} |\hat{\beta}_\omega(w_n)|}$ (about 10^{-4}) is very small, i.e. the noise in the samples creates huge error in generating the Fourier transforms.

Solution: A stable band-pass kernel:

- Multiplication causes problems
- Consider the summation instead, i.e.

$$\hat{\psi}_{\boldsymbol{\omega}}(w) = \hat{\beta}_{\boldsymbol{\omega}^+}(w) + \hat{\beta}_{\boldsymbol{\omega}^-}(w). \tag{9}$$

Problems! The Strang-fix conditions are not satisfied.

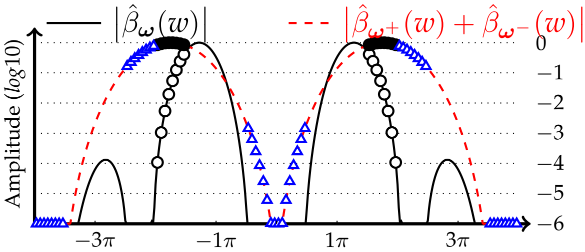
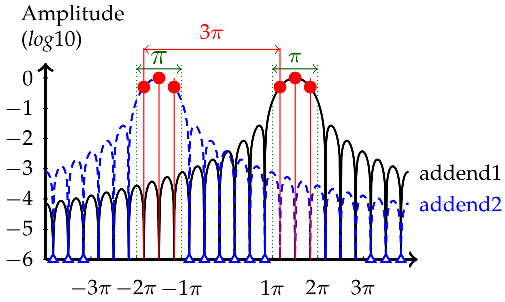


Figure: Black dots mark $|\hat{\beta}_{\boldsymbol{\omega}^+}(w) + \hat{\beta}_{\boldsymbol{\omega}^-}(w)|$ at frequencies $\boldsymbol{\omega}^+$ and $\boldsymbol{\omega}^-$. Blue triangles shows the frequency response at frequencies $\boldsymbol{\omega}^\pm \pm 4\pi$ and $\boldsymbol{\omega}^\pm \pm 2\pi$. The Strang-Fix conditions on $\boldsymbol{\omega}$ are not satisfied after summation. $\boldsymbol{\omega}^- + 4\pi$ are located in the main lobe of $\hat{\beta}_{\boldsymbol{\omega}^+}(w)$

A STABLE BAND-PASS KERNEL: *Solution- a simplified example*



- shrink the main lobe so that $\omega^- + 2\pi l$ is not in the main lobe of ω^+ for any l
- carefully choose parameters in ω^+, so that the zeros of the two addends align.

A STABLE BAND-PASS KERNEL: *dilated E-spline*

The dilated E-spline is defined as:

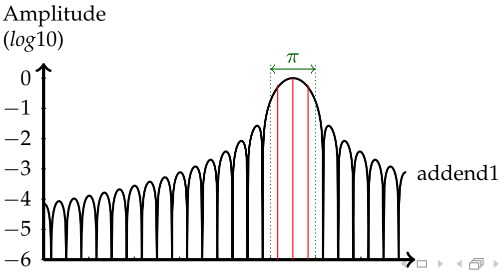
Dilation

$$\hat{\beta}_{\omega,2M}(w) = \prod_{n=0}^{P} \frac{1 - e^{2Mj(w_n - w)}}{2Mj(w - w_n)}, \qquad (10)$$

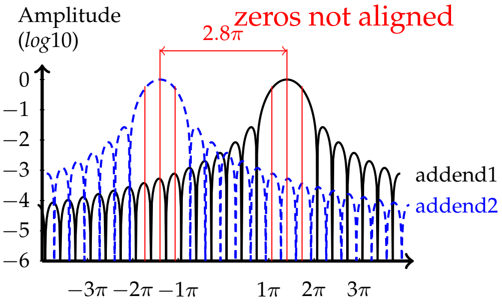
which satisfies

$$\hat{\beta}_{\boldsymbol{\omega},2M}(w_n) \neq 0, \hat{\beta}_{\boldsymbol{\omega},2M}(w_n + l\frac{\pi}{M}) = 0, \forall l \in \mathbb{Z} \setminus 0. \qquad (11)$$

And its main lobe size is $\frac{2\pi}{2M}$.

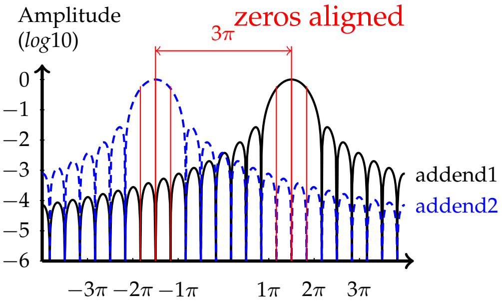


A STABLE BAND-PASS KERNEL: *constrains of the parameters*



- The center of the lobe must be at multiple of $\frac{\pi}{M}$
- The parameters are symmetrical to the center

A STABLE BAND-PASS KERNEL: *constrains of the parameters*



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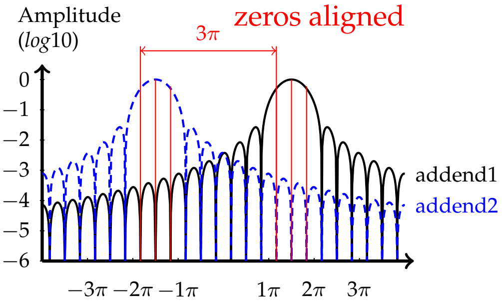
A STABLE BAND-PASS KERNEL: *constrains of the parameters*



- The center of the lobe must be at multiple of $\frac{\pi}{M}$
- The parameters are symmetrical to the center

A band-pass E-spline:

Theorem 1: constructing a stable band-pass real E-spline

Let

$$\hat{\psi}_{\boldsymbol{\omega},2M}(w) = \left(\hat{\beta}_{\boldsymbol{\omega},2M}(w) + \hat{\beta}_{-\boldsymbol{\omega},2M}(w)\right), \tag{12}$$

where M is a positive integer. Assume

$$\begin{aligned} & w_0 \leq w_1 \leq \cdots \leq w_P, w_P - w_0 < \frac{\pi}{M}, \\ & w_n + w_{P-n} = k\frac{\pi}{M}, k \in \mathbb{N} \setminus \{2Ml \big| l \in \mathbb{N}.\} \end{aligned} \tag{13}$$

Then $\hat{\psi}_{\boldsymbol{\omega},2M}(w)$ is able to reproduce exponential $e^{\pm j w_n t}$, and is a real function:

$$\psi_{\boldsymbol{\omega},2M}(t) = 2\beta_{\boldsymbol{\omega}-\frac{k\pi}{2M},2M}(t) \cos\left(\frac{k\pi}{2M}t\right). \tag{14}$$

Multiple-subband E-splines

- Extension of the band-bass E-spline: for some real b_i,

$$\phi^M_{\boldsymbol{\omega}_0,K_0}(t) = \beta_{\boldsymbol{\omega}_0,2M}(t)\left(\sum_{i=1}^{M} 2b_i \cos\left(\frac{2k_i+K_0}{2M}\pi t\right)\right), \quad (15)$$

where $\beta_{\boldsymbol{\omega}_0,2M}(t)$ is a real dilated E-spline.

Theorem 2: constructing a MEMS

Let $r_i = rem(k_i, 2M)$. $K_0 \in \{0,1\}$. MEMS $\phi^M_{\boldsymbol{\omega},K_0}(t)$ can reproduce exponential $e^{\pm j w_{n,i} t}$ if

$$r_i + r_l \neq 2M - K_0 \text{ and } \min_{i \neq l} |r_i - r_l| \neq 0. \quad (16)$$

Here $w_{n,i} = w_n + \frac{2k_i+K_0}{2M}\pi$.

MULTIPLE-SUBBAND E-SPLINE: Example

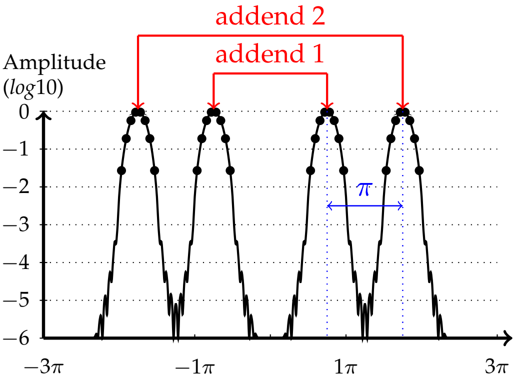


Figure: The frequency amplitude of MEMS, where $M = 2$, $P = 7$, $\boldsymbol{\omega}_0 = \{\frac{k}{32} | k = -7 : 2 : 7\}$, $K_0 = 1$, $k_1 = 1$, $k_2 = 3$. $\left|\hat{\phi}^{M}_{\boldsymbol{\omega}_0, K_0}(w_n + \frac{2k_i + K_0}{4}\pi)\right|$ is marked by the solid dot.

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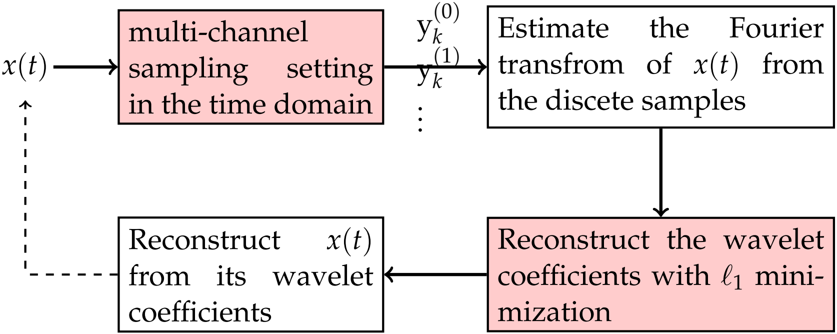


ℓ_1 NORM MINIMIZATION

- The relationship between Fourier transform of $x(t)$ and its wavelet coefficients:

wavelet coefficients

$$\hat{x}(w) = \boldsymbol{\gamma}^T(w)\boldsymbol{\eta}. \tag{17}$$

- $\boldsymbol{\eta}$ can be recovered from Fourier transforms of $x(t)$ at set Ω by

$$\begin{aligned} &\min |\boldsymbol{\eta}|_1 \\ &s.t. \ \ \|\hat{x}(\Omega) - \boldsymbol{\gamma}(\Omega)\boldsymbol{\eta}\|_2 \leq \sigma, \end{aligned} \tag{18}$$

given that Ω (f-pattern) is randomly picked from $[-N_0, N_0]/2\tau$ and with sufficient size. N_0 is a constant dependent on the signal, and τ is the support of $x(t)$.

A MULTICHANNEL SCHEME

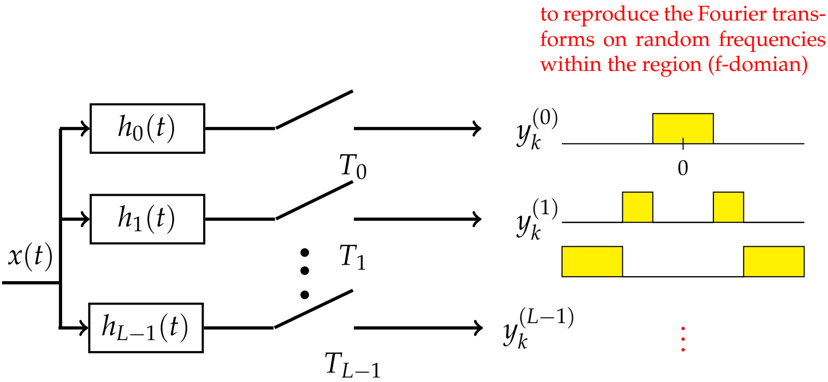


Figure: The multichannel sampling scheme. Channels are used to estimate the Fourier transform of $x(t)$ at frequencies from different regions in the frequence domain. By choosing different bandwidth of the region and T_l, an variable density sampling pattern is obtained.

THE FILTERS

The filters are generated by the following parameters

- $$\boldsymbol{\omega}_0 = \{\frac{2n+1}{2P+2}\frac{\pi}{M_l} | n = 0 : P+1\}$$
  $$K_0 = 0$$
  where P and M_l are picked manually,
- k_i are randomly picked from a specific region such that $\omega_{n,i}/T_l$ is within the predefined frequency range.
- $\{k_i\}$ have to satisfy Theorem 2.

THE RESULTED F-PATTERN

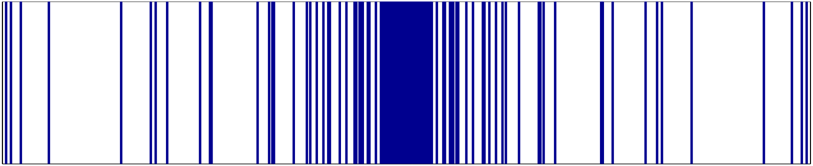


Figure: The resulted pattern (Ω of (18)) in f-domain using the filter setting describe in the previous slide. 0 frequency in the center. The f-pattern is the result of picking k_i satisfying Theorem 2 uniformly from a given range. The output of the multichannel system can be used to estimate the Fourier transform of $x(t)$ on the f-pattern. The f-pattern will be used in the following numerical example.

It is much easier to randomly pick k_i that generates random f-pattern than to calculate k_i, M_l and P according to a given f-pattern

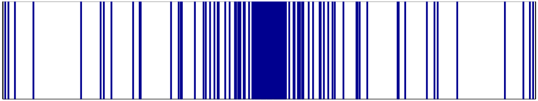
Why this multi-channel setting?

- additional degrees of freedom
- reduced complexity in setting parameters
- sampling pattern with variable density
- easy to modify (add or delete channels)

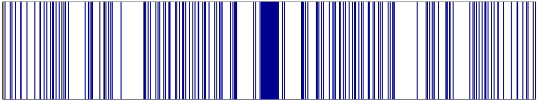
NUMERICAL EXAMPLE: sparse on db4 with 13 non-zeros

Figure: The pulse. The number of non-zeros in each level is [1 2 4 2 1 1 1 1].

THE F-PATTERN

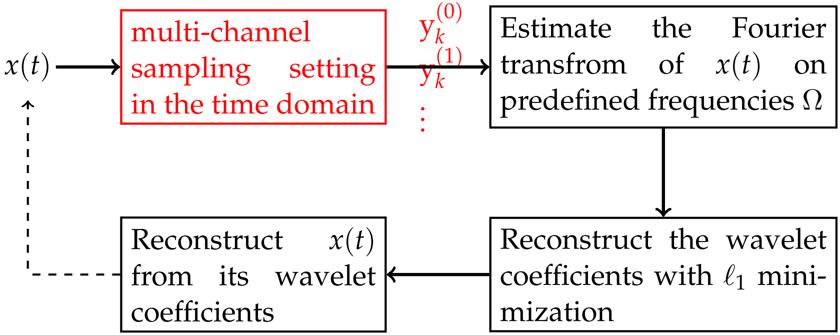


The f-pattern of the proposed multichannel setting



benchmark(uniformly random)

SAMPLING IS IN SPATIAL DOMAIN



NUMERICAL EXAMPLE: sparse on db4 with 13 non-zeros

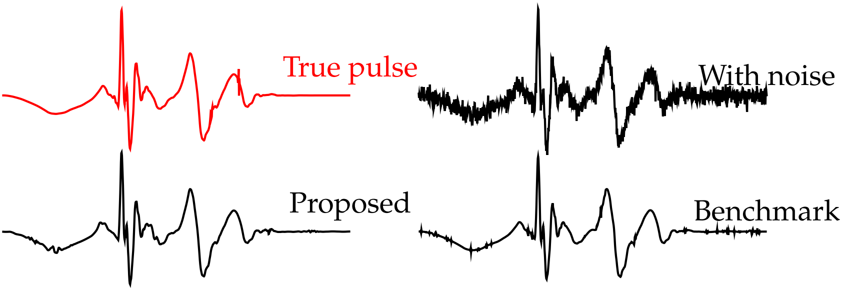


Figure: Example performance under the noisy condition (10dB). $N_T = 32, N_0 = 1024$.

NUMERICAL EXAMPLE: sparse on db4 with 13 non-zeros

Table: The recovery error of different designs in the presence of noise. The average performance of 100 realizations.

	50dB	40dB	30dB	20dB	10dB
Proposed	0.0025	0.0078	0.0222	0.0591	0.1082
Benchmark	0.0025	0.0080	0.0245	0.0679	0.1277

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Conclusion

- A new method for reconstructing exponential reproducing kernels
- The constructed kernel provides us a way to easily obtain the Fourier transforms on frequencies over a wide range.
- A multichannel sampling strategy to sample wavelet-sparse signal with variable density in the frequency domain.
- ℓ_1 minimization reconstructs the signal to satisfactory level.

QUESTIONS?

Thank you!