# Additive Link Metrics Identification: Proof of Selected Lemmas and Propositions

Liang Ma†, Ting He‡, Kin K. Leung†, Ananthram Swami§, and Don Towsley*
†Imperial College, London, UK. Email: {l.ma10, kin.leung}@imperial.ac.uk
‡IBM T. J. Watson Research Center, Hawthorne, NY, USA. Email: the@us.ibm.com
§Army Research Laboratory, Adelphi, MD, USA. Email: ananthram.swami.civ@mail.mil
*University of Massachusetts, Amherst, MA, USA. Email: towsley@cs.umass.edu

## I. Introduction

Selected lemmas and propositions in [1] are proved in detail in this report. We first list the lemmas and propositions in Section II and then give the corresponding proofs in Section III. See the original paper [1] for terms and definitions.

## II. Lemmas and Propositions

Let $\mathcal{H}$ denote the interior graph of graph $\mathcal{G}$, where two monitoring nodes ($m_1$ and $m_2$) are employed. In this report, Conditions ① and ② refer to the two following conditions.

① $\mathcal{G}-l$ is 2-edge-connected for each interior link $l$ in $\mathcal{H}$;
② $\mathcal{G}+m_1m_2$ is 3-vertex-connected.

**Lemma II.1.** When the interior graph $\mathcal{H}$ of $\mathcal{G}$ is connected, the corresponding measurement matrix $\mathbf{R}$ can be linearly transformed to

$$\begin{array}{c} \begin{array}{ccc} W_{m_1a_1}\cdots W_{m_1a_{k_1}} & W_{b_1m_2}\cdots W_{b_{k_2}m_2} & W_{l_1}\cdots W_{l_{k_h}} \end{array} \\ \left(\begin{array}{cccc|cccc|c} 1 & & & & 1 & & & & \\ 1 & & & & & 1 & & & \\ \vdots & & & & & & \ddots & & \mathbf{B} \\ 1 & & & & & & & 1 & \\ \hline -1 & 1 & & & & & & & \\ -1 & & 1 & & & & & & \mathbf{T} \\ \vdots & & & \ddots & & & & & \\ -1 & & & 1 & & & & & \\ \hline & & & & & & & & \mathbf{L} \end{array}\right) \end{array}, \tag{1}$$

where the rest entries are zero, $\mathbf{B}$ is a $k_2 \times k_h$ Boolean matrix[1], $\mathbf{T}$ is a $(-1,0,1)$-Matrix[2] with dimensions $(k_1-1)\times k_h$, and $\mathbf{L}$ with $k_h$ columns is a matrix associated with all rows in the restructured measurement matrix $\mathbf{R}$ involving only $(W_{l_i})_{i=1}^{k_h}$.

**Proposition II.2.** For graph $\mathcal{G}$, if all link metrics in the associated interior graph are identifiable through simple path measurements, then $\mathcal{G}+m_1m_2$ is 3-vertex-connected.

**Proposition II.3.** Using two monitoring nodes, the necessary and sufficient condition for $\mathcal{G}+m_1m_2$ being a 3-vertex-connected graph is when 2 nodes are deleted in $\mathcal{G}$, the remaining graph is still connected, *or* every connected component each has a monitoring node.

[1] *Boolean matrix* is a matrix each of whose entries is 0 or 1.
[2] *$(-1,0,1)$-Matrix* is a matrix each of whose entries is $-1$, 0, or 1.

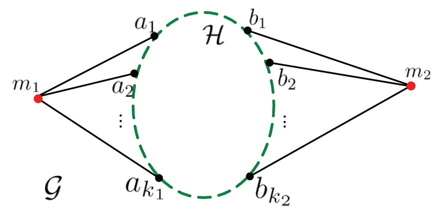


Fig. 1. Reorganized graph $\mathcal{G}$, where $a_i$ and $b_j$ can be the same node.

**Lemma II.4.** If graph $\mathcal{G}$ satisfies Conditions ① and ②, for any link $vw$ in the interior graph of $\mathcal{G}$, two cycles ($\mathcal{C}_1$ and $\mathcal{C}_2$) can be discovered in $\mathcal{G}$, such that

(a) $\mathcal{C}_1$ is a face[3];
(b) $vw$ is the only common link between $\mathcal{C}_1$ and $\mathcal{C}_2$;
(c) $\mathcal{C}_1$ and $\mathcal{C}_2$ have *one* common node at most, apart from $v$ and $w$;
(d) there exists path $\mathcal{P}_1$ connecting[4] $m_1^*$ and a node on $\mathcal{C}_1 - v - w$ and $\mathcal{P}_2$ connecting $m_2^*$ and a node on $\mathcal{C}_2 - v - w$;
(e) $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$;
(f) $L(\mathcal{P}_1)\cap L(\mathcal{C}_1 - v - w) = \emptyset$, $L(\mathcal{P}_2)\cap L(\mathcal{C}_2 - v - w) = \emptyset$;
(g) $v, w \notin V(\mathcal{P}_1)$ and $v, w \notin V(\mathcal{P}_2)$.

**Proposition II.5.** If graph $\mathcal{G}$ satisfies Conditions ① and ②, then

(a) for any face in graph $\mathcal{G}$, there is at most *one* border-link in this face;
(b) for any border-link $vw$ in the interior graph of $\mathcal{G}$, it can discover a face without traversing $m_1$ and $m_2$;
(c) for every border-link $vw$ on face $\mathcal{C}_1$ with $L(\mathcal{C}_1) \subseteq L(\mathcal{H})$, there exist paths $\mathcal{P}(m_1, v)$ and $\mathcal{P}(m_2, w)$ with $\mathcal{P}(m_1, v)\cap \mathcal{P}(m_2, w) = \emptyset$, $\mathcal{P}(m_1, v)\mathring{v} \cap \mathcal{C}_1 = \emptyset$ and $\mathcal{P}(m_2, w)\mathring{w} \cap \mathcal{C}_1 = \emptyset$.

## III. Proofs

### A. Proof of Lemma II.1

Suppose that the two monitoring nodes $m_1$ and $m_2$ give rise to a connected interior graph $\mathcal{H}$ in $\mathcal{G}$ (Fig. 1). Let $(l_i)_{i=1}^{k_h} := L(\mathcal{H})$ be the link set of $\mathcal{H}$ with $k_h := ||\mathcal{H}||$. Consider the measurement matrix $\mathbf{R}_{\mathbf{a_i}}$ ($i \in \{1, \cdots, k_1\}$) corresponding to all possible paths $m_1 \to a_i \to \ldots \to b_j \to m_2$ ($j = 1, \ldots, k_2$). Then $\mathbf{R}_{\mathbf{a_i}}$ is of the form

[3] In the sequel, $\mathcal{C}_1$ means a cycle as well as a face.
[4] Let $m_1^*, m_2^* \in \{m_1, m_2\}$ with $m_1^* \neq m_2^*$.

$$\mathbf{R_{a_i}} = \left(\begin{array}{c|ccc|c}
W_{m_1a_i} & W_{b_1m_2} & \cdots & W_{b_{k_2}m_2} & W_{l_1}\cdots W_{l_{k_h}} \\
1 & 1 & & & \\
\vdots & \vdots & & & \mathbf{B_{i1}} \\
1 & 1 & & & \\ \hline
1 & & 1 & & \\
\vdots & & \vdots & & \mathbf{B_{i2}} \\
1 & & 1 & & \\ \hline
\vdots & & \ddots & & \vdots \\ \hline
1 & & & 1 & \\
\vdots & & & \vdots & \mathbf{B_{ik_2}} \\
1 & & & 1 &
\end{array}\right),$$

where $\mathbf{B_{ij}}$ ($j = 1, \ldots, k_2$) with $k_h$ columns is the Boolean matrix corresponding to all simple paths between $a_i$ and $b_j$ in $\mathcal{H}$ (i.e., the $(p,\ q)$-th entry indicates whether link $l_q$ in $\mathcal{H}$ appears on the $p$th path between $a_i$ and $b_j$). Note that at least one such path exists, i.e., $\mathbf{B_{ij}}$ is nonempty, since $\mathcal{H}$ is connected. Among all rows containing $W_{m_1a_i}$ and $W_{b_jm_2}$ in $\mathbf{R_{a_i}}$, if we subtract the first row from the others, then the other rows are only non-zero in entries corresponding to $(W_{l_i})_{i=1}^{k_h}$. Reorganizing these subtracted rows to the bottom, we transform $\mathbf{R_{a_i}}$ to

$$\mathbf{R'_{a_i}} = \left(\begin{array}{c|ccc|c}
W_{m_1a_i} & W_{b_1m_2} & \cdots & W_{b_{k_2}m_2} & W_{l_1}\cdots W_{l_{k_h}} \\
1 & 1 & & & \mathbf{r_{i1}} \\
1 & & 1 & & \mathbf{r_{i2}} \\
\vdots & & \ddots & & \vdots \\
1 & & & 1 & \mathbf{r_{ik_2}} \\ \hline
 & & & & \mathbf{L_{a_i}}
\end{array}\right), \qquad (2)$$

where $\mathbf{r_{ij}}$ ($j = 1, \ldots, k_2$) is the first row of $\mathbf{B_{ij}}$, and $\mathbf{L_{a_i}}$ is a matrix derived from the subtraction operation. Combining all $\mathbf{R'_{a_i}}$ $\big(i \in \{1, \cdots, k_1\}\big)$ in (2), the restructured measurement matrix $\mathbf{R}$ is

$$\left(\begin{array}{cccc|cccc|c}
\multicolumn{4}{c|}{W_{m_1a_1}\cdots W_{m_1a_{k_1}}} & \multicolumn{4}{c|}{W_{b_1m_2}\cdots W_{b_{k_2}m_2}} & W_{l_1}\cdots W_{l_{k_h}} \\
1 & & & & 1 & & & & \mathbf{r_{11}} \\
1 & & & & & 1 & & & \mathbf{r_{12}} \\
\vdots & & & & & & \ddots & & \vdots \\
1 & & & & & & & 1 & \mathbf{r_{1k_2}} \\ \hline
 & 1 & & & 1 & & & & \mathbf{r_{21}} \\
 & 1 & & & & 1 & & & \mathbf{r_{22}} \\
 & \vdots & & & & & \ddots & & \vdots \\
 & 1 & & & & & & 1 & \mathbf{r_{2k_2}} \\ \hline
 & & \ddots & & & & \vdots & & \vdots \\ \hline
 & & & 1 & 1 & & & & \mathbf{r_{k_11}} \\
 & & & 1 & & 1 & & & \mathbf{r_{k_12}} \\
 & & & \vdots & & & \ddots & & \vdots \\
 & & & 1 & & & & 1 & \mathbf{r_{k_1k_2}} \\ \hline
 & & & & & & & & \mathbf{L_1}
\end{array}\right), \qquad (3)$$

where $\mathbf{L_1}$ is the matrix formed by arranging $\mathbf{L_{a_1}}, \ldots, \mathbf{L_{a_{k_1}}}$ vertically. We apply the following linear transformations to (3): (i) first, subtracting row $i$ from row $qk_2 + i$ for each $q = 1, \ldots, k_1 - 1$ and $i = 1, \ldots, k_2$; (ii) then, subtracting row $qk_2 + 1$ from row $qk_2 + i$ for each $q = 1, \ldots, k_1 - 1$ and $i = 2, \ldots, k_2$; (iii) finally, moving all rows containing $r_{ij} - r_{1j} - r_{i1}$ ($i = 2, \cdots, k_1$ and $j = 2, \cdots, k_2$) to the matrix bottom. Ignoring entries whose values are zeros ($(W_{m_1a_i})_{i=1}^{k_1}$ and $(W_{b_jm_2})_{j=1}^{k_2}$ are not involved in the rows containing $r_{ij} - r_{1j} - r_{i1}$ ($i = 2, \cdots, k_1$ and $j = 2, \cdots, k_2$)), (3) is transformed into

$$\left(\begin{array}{cccc|cccc|c}
\multicolumn{4}{c|}{W_{m_1a_1}\cdots W_{m_1a_{k_1}}} & \multicolumn{4}{c|}{W_{b_1m_2}\cdots W_{b_{k_2}m_2}} & W_{l_1}\cdots W_{l_{k_h}} \\
1 & & & & 1 & & & & \mathbf{r_{11}} \\
1 & & & & & 1 & & & \mathbf{r_{12}} \\
\vdots & & & & & & \ddots & & \vdots \\
1 & & & & & & & 1 & \mathbf{r_{1k_2}} \\ \hline
-1 & 1 & & & & & & & \mathbf{r_{21} - r_{11}} \\
-1 & & 1 & & & & & & \mathbf{r_{31} - r_{11}} \\
\vdots & & & \ddots & & & & & \vdots \\
-1 & & & 1 & & & & & \mathbf{r_{k_11} - r_{11}} \\ \hline
 & & & & & & & & \mathbf{L}
\end{array}\right), \qquad (4)$$

where

$$\mathbf{L} := \left(\begin{array}{c}
W_{l_1}\ W_{l_2}\cdots\ W_{l_{k_h}} \\
\mathbf{L_1} \\
\mathbf{r_{22} - r_{12} - r_{21}} \\
\mathbf{r_{23} - r_{13} - r_{21}} \\
\vdots \\
\mathbf{r_{2k_2} - r_{1k_2} - r_{21}} \\
\mathbf{r_{32} - r_{12} - r_{31}} \\
\mathbf{r_{33} - r_{13} - r_{31}} \\
\vdots \\
\mathbf{r_{3k_2} - r_{1k_2} - r_{31}} \\
\vdots \\
\mathbf{r_{k_12} - r_{12} - r_{k_11}} \\
\mathbf{r_{k_13} - r_{13} - r_{k_11}} \\
\vdots \\
\mathbf{r_{k_1k_2} - r_{1k_2} - r_{k_11}}
\end{array}\right).$$

Entries in $\mathbf{r_{i1} - r_{11}}$ are of the value of -1, 0, or 1, since each entry in $\mathbf{r_{ij}}$ is 0 or 1. Therefore, $\mathbf{B} := \begin{pmatrix} \mathbf{r_{11}} \\ \mathbf{r_{12}} \\ \vdots \\ \mathbf{r_{1k_2}} \end{pmatrix}$ is a $k_2 \times k_h$ Boolean matrix, while $\mathbf{T} := \begin{pmatrix} \mathbf{r_{21} - r_{11}} \\ \mathbf{r_{31} - r_{11}} \\ \vdots \\ \mathbf{r_{k_11} - r_{11}} \end{pmatrix}$ is a $(-1, 0, 1)$-Matrix with dimensions $(k_1 - 1) \times k_h$. Consequently, when the interior graph $\mathcal{H}$ of $\mathcal{G}$ is connected, the corresponding measurement matrix $\mathbf{R}$ can be linearly transformed to

$$\left(\begin{array}{cccc|cccc|c}
\multicolumn{4}{c|}{W_{m_1a_1}\cdots W_{m_1a_{k_1}}} & \multicolumn{4}{c|}{W_{b_1m_2}\cdots W_{b_{k_2}m_2}} & W_{l_1}\cdots W_{l_{k_h}} \\
1 & & & & 1 & & & & \\
1 & & & & & 1 & & & \mathbf{B} \\
\vdots & & & & & & \ddots & & \\
1 & & & & & & & 1 & \\ \hline
-1 & 1 & & & & & & & \\
-1 & & 1 & & & & & & \mathbf{T} \\
\vdots & & & \ddots & & & & & \\
-1 & & & 1 & & & & & \\ \hline
 & & & & & & & & \mathbf{L}
\end{array}\right). \blacksquare$$

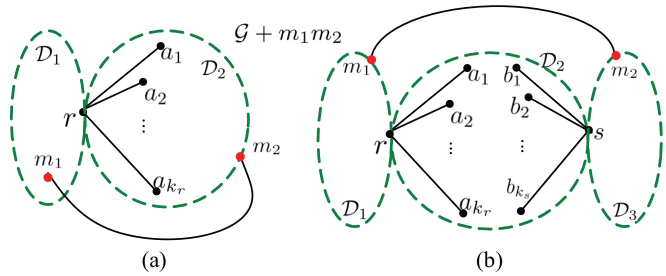


Fig. 2. $\mathcal{G}+m_1m_2$ is 3-vertex-connected.

### *B. Proof of Proposition II.2*

Suppose interior graph $\mathcal{H}$ of $\mathcal{G}$ is identifiable and $\mathcal{G}+m_1m_2$ is not 3-vertex-connected, then the *connectivity*[5] of $\mathcal{G}+m_1m_2$ is 1 or 2.

*1)* Consider the case that the connectivity of $\mathcal{G}+m_1m_2$ is 1 and node $r \in V(\mathcal{G}+m_1m_2)$ is deleted. According to the assumption of $\mathcal{H}$ being a connected graph, $\mathcal{G}$ must be 1-vertex-connected. First, if $r$ is a monitoring node, the resulting $\mathcal{G}+m_1m_2-r=\mathcal{G}-r$ is not disconnected since $\mathcal{H}$ is connected. Second, we assume $\mathcal{G}$ is separated into two components, denoted by $\mathcal{D}_1$ and $\mathcal{D}_2$, after deleting $r$ ($r \in V(\mathcal{H})$). If each of $\mathcal{D}_1$ and $\mathcal{D}_2$ contains a monitoring node, then link $m_1m_2$ connects $\mathcal{D}_1$ and $\mathcal{D}_2$ again. If one of $\mathcal{D}_1$ and $\mathcal{D}_2$, say $\mathcal{D}_1$, does not have monitoring nodes in it, then all $m_1 \to m_2$ paths employing links in $\mathcal{D}_1$ must both enter and leave $\mathcal{D}_1$ at $r$, thus forming a cycle in this path construction. Therefore, the connectivity of graph $\mathcal{G}+m_1m_2$ must be greater than 1.

*2)* Now we assume the connectivity of $\mathcal{G}+m_1m_2$ is 2. First, Consider deleting one monitoring node and a non-monitoring node (displayed in Fig. 2-a). If deleting $r$ ($r \in V(\mathcal{H})$) and $m_2$ results in $\mathcal{D}_2$ being separated, then $\mathcal{D}_1$ and $\mathcal{D}_2$ only have one common node, denoted by $r$. In this case, $m_1$ must be in $\mathcal{D}_1$; otherwise, a cycle is formed when using links in $\mathcal{D}_1$ to construct $m_1 \to m_2$ paths. However, even if all link metrics in $\mathcal{D}_1$ have been identified, $(W_{ra_i})_{i=1}^{k_r}$ ($a_i \neq m_2$ and $k_r \geq 2$, since $\mathcal{G}$ is proved to 2-edge-connected in [1] when its interior graph is identifiable) are uncomputable, according to Corollary III.2 in [1]. This contradicts the claim that $\mathcal{H}$ is identifiable. Second, if $m_1$ and $m_2$ are deleted, the remaining graph is $\mathcal{H}$, which is connected, according to the assumption that the interior graph of $\mathcal{G}$ is connected. Third, now we consider deleting $r$ and $s$ ($r,s \in V(\mathcal{H})$, $r \neq s \neq m_1 \neq m_2$). If $\mathcal{G}$ is separated and each component has a monitoring node, then $m_1m_2$ can connect these two components again. If the separated component does not have monitoring nodes in it (such as $\mathcal{D}_2$ in Fig. 2-b), then according to Corollary III.2 in [1], $(W_{ra_i})_{i=1}^{k_r}$ and $(W_{sb_i})_{i=1}^{k_s}$ are uncomputable even if all link metrics in $\mathcal{D}_1$ and $\mathcal{D}_3$ have been identified. This also contradicts the claim that $\mathcal{H}$ is identifiable.

Thus, the connectivity of $\mathcal{G}+m_1m_2$ is greater than 2. Therefore, $\mathcal{G}+m_1m_2$ is 3-vertex-connected when its interior graph is identifiable. ■

[5] The greatest integer $k$ such that $\mathcal{G}$ is $k$-vertex-connected is the *connectivity* of $\mathcal{G}$.

### *C. Proof of Proposition II.3*

*Necessary part.*

*1)* If $\mathcal{G}$ is separated by deleting 2 non-monitoring nodes, then each component must have a monitoring node; otherwise, $\mathcal{G}+m_1m_2$ is 2-vertex-connected.

*2)* If one of these deleted 2 nodes is a monitoring node, say $m_1$, then $m_1m_2$ is deleted as well. Deleting any other node except $m_2$ will not result in the separation of G. If separated, $\mathcal{G}+m_1m_2$ is 2-vertex-connected.

*3)* If $m_1$ and $m_2$ are deleted, we can obtain sub-graph $\mathcal{H}$, which is connected according to the assumption.

*Sufficient part.*

*1)* When 2 nodes are deleted in $\mathcal{G}$, if it remains connected, then $\mathcal{G}$ is 3-vertex-connected, so is $\mathcal{G}+m_1m_2$.

*2)* If $\mathcal{G}$ is separated after deleting two nodes and each separated component has a monitoring node, then these components are connected again by link $m_1m_2$. Therefore, $\mathcal{G}+m_1m_2$ is 3-vertex-connected. ■

### *D. Proof of Lemma II.4*

*1)* For $vw \in L(\mathcal{H})$, an H-path[6] $\mathcal{P}_1$ from $v$ to $w$ can be discovered for a 2-vertex-connected graph, according to Proposition 3.1.3 [2] ($\mathcal{G}$ is a 2-vertex-connected graph, since $\mathcal{G}+m_1m_2$ is 3-vertex-connected), then a cycle $\mathcal{C}'_1=\mathcal{P}_1+vw$ is formed. If $xy \in L(\mathcal{G})$ with $x,y \in V(\mathcal{C}'_1)$ and $xy \notin L(\mathcal{C}'_1)$, then use $xy$ to replace $\mathcal{P}_{\mathcal{C}'_1}(x,y)$ recursively, i.e., $\mathcal{C}'_1 = \mathcal{C}'_1 \setminus \mathring{\mathcal{P}}_{\mathcal{C}'_1}(x,y)+xy$, until no such $xy$ exists, where $\mathcal{P}_{\mathcal{C}'_1}(x,y)$ is the path from $x$ to $y$ in $\mathcal{C}'_1$ with $vw \notin L(\mathcal{P}_{\mathcal{C}'_1}(x,y))$. Finally, $\mathcal{C}''_1=\mathcal{C}'_1$ is an induced cycle. If $\mathcal{C}''_1$ is not a face, then there exists separated component $\mathcal{D}$ ($\mathcal{D}\cap\{m_1,m_2\}=\emptyset$, when $\mathcal{C}''_1$ is deleted), in which *all* paths from $V(\mathcal{D})$ to $m_1$ or $m_2$ must have three or more (since $\mathcal{G}+m_1m_2$ is 3-vertex-connected) common nodes with $\mathcal{C}''_1$. For any $v_1 \in V(\mathcal{D})$, degree[7] $d(v_1) \geq 2$, since $\mathcal{G}$ satisfies Condition ①. Furthermore, there must be an inner path $\mathcal{P}_{in}(x_1,x_2)$ incident with $x_1$ and $x_2$ ($x_1,x_2 \in V(\mathcal{C}''_1)$) and an interior node $v_2 \in V(\mathcal{D})$ with $v_2 \in V\big(\mathcal{P}_{in}(x_1,x_2)\big)$. Using $\mathcal{P}_{in}(x_1,x_2)$ to replace $\mathcal{P}_{\mathcal{C}''_1}(x_1,x_2)$ ($vw \notin L(\mathcal{P}_{\mathcal{C}''_1}(x_1,x_2))$) in $\mathcal{C}''_1$ recursively, i.e., $\mathcal{C}''_1 = \mathcal{C}''_1 \setminus \mathring{\mathcal{P}}_{\mathcal{C}''_1}(x_1,x_2)\bigcup\mathcal{P}_{in}(x_1,x_2)$, until no such $\mathcal{P}_{in}(x_1,x_2)$ exists. As a result, a face $\mathcal{C}_1=\mathcal{C}''_1$ can be discovered in $\mathcal{G}$.

*2)* Suppose $\mathcal{C}_2$ satisfying (b) in Lemma II.4 cannot be discovered, then $\mathcal{C}_2$ must share some common links with $\mathcal{C}_1\setminus\{vw\}$. Let $rs$ be one of these common links, then if $vw$ is deleted, all possible paths connecting $v$ and $w$ must traverse link $rs$. In this case, $rs$ becomes a bridge, contradicting Condition ①.

*3)* Suppose there are always two common nodes no matter what strategy is used to select the two cycles. Let $r,s \in$

[6] $\mathcal{P}$ ($||\mathcal{P}|| \geq 1$) is an *H-path* of graph $\mathcal{H}$ if $\mathcal{P}$ meets $\mathcal{H}$ exactly in its end nodes.

[7] The *degree* of node $v$ is the number of links incident with $v$, denoted by $d(v)$.

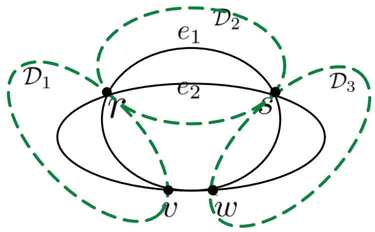


Fig. 3. Two cycles with two common nodes.

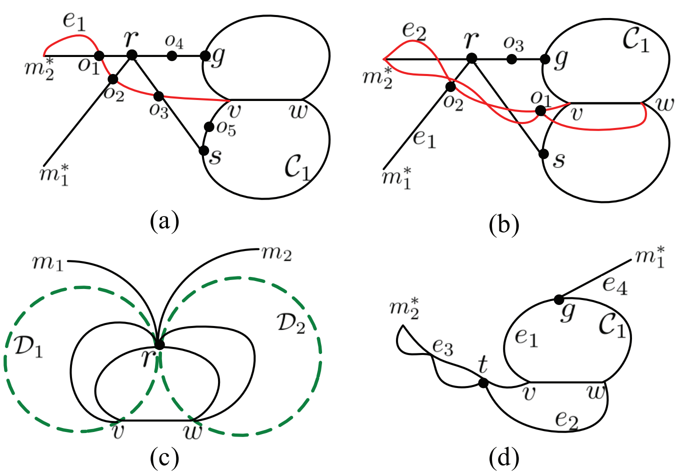


Fig. 4. Construction of two cycles.

$V(\mathcal{C}_1) \cap V(\mathcal{C}_2) \setminus \{v, m\}$, the two paths connecting $r$ and $s$ in $\mathcal{D}_2$ are $r\underline{e_1}s$ and $r\underline{e_2}s$ (shown in Fig. 3). It has been proved that $r\underline{e_1}s$ and $r\underline{e_2}s$ cannot have common links, thus $\mathring{r}\underline{e_1}\mathring{s} \neq \mathring{r}\underline{e_2}\mathring{s}$. If $vw$ is deleted, any $v \to w$ paths must first traverse $r$ and then traverse $s$. Therefore, $\mathcal{G}$ is composed of three components ($\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$) and link $vw$ with $V(\mathcal{D}_1 \cap \mathcal{D}_2) = \{r\}$, $V(\mathcal{D}_2 \cap \mathcal{D}_3) = \{s\}$ and $V(\mathcal{D}_1 \cap \mathcal{D}_3) = \emptyset$. For $\mathcal{D}_2$, $|V(r\underline{e_1}s \cup r\underline{e_2}s) \setminus \{r, s\}| \geq 1$, since $\mathring{r}\underline{e_1}\mathring{s} \neq \mathring{r}\underline{e_2}\mathring{s}$ and $||(\mathcal{C}_1 - vw) \cap (\mathcal{C}_2 - vw)|| = 0$. Similarly, $|\mathcal{D}_1 - r - v| \geq 1$ and $|\mathcal{D}_3 - s - w| \geq 1$. Only two of these three components can have $m_1$ or $m_2$. Thus, the third component without monitoring nodes is separated when the two common nodes with adjacent components are deleted, contradicting Lemma II.3.

*4)* $\mathcal{G}$ is connected; therefore, (d) in Lemma II.4 is true.

*5)* (i). If all $\mathcal{P}_1$ ($m_1^* \in V(\mathcal{P}_1)$) must traverse $m_2^*$, then $m_2^*$ is a cutvertex[8] in $\mathcal{G}$, contradicting Lemma II.3.

(ii). Let $\mathcal{C}_1 = v\underline{s}w + vw$. For any $\mathcal{P}_1$ and $\mathcal{P}_2$, if they must have a common node, say $r$ (see Fig. 4-a), then $r$ cannot be a cutvertex, because $\mathcal{G}$ is 2-vertex-connected. Therefore, there must be another path employing $v$ or $w$, say $m_2^*\underline{o_1 \cdots o_5 v}g$ ($r \notin V(m_2^*\underline{o_1 \cdots o_5 v}a)$), to connect $m_2^*$ and $g$. $m_2^*\underline{o_1 \cdots o_5 v}g$ might have common nodes $(o_1, \cdots, o_5)$ with other paths. However, if $o_4$ or $o_5$ is the common node, then $\mathcal{P}_1$ and $\mathcal{P}_2$ do not need to traverse $r$ to connect the two cycles. If $m_2^*\underline{o_1 \cdots o_5 v}a$ must traverse $(o_i)_{i=1}^3$, then $m_2^*$ cannot connect to the two cycles when $r$ and $o_i$ are deleted, contradicting Lemma II.3. Thus, an $m_2^*\underline{o_1}v$ which does not have unavoidable common nodes $o_1$, $o_2$ and $o_3$ can be constructed. Therefore, $\mathcal{C}_2$ can be reselected, i.e., $\mathcal{C}_2 = v\underline{o_1 r o_4 g}w + vw$ with $\mathcal{P}_2 = m_2^*\underline{e_1}o_1$ and $\mathcal{P}_1 = m_1^*\underline{r}s$.

(iii). Let $\mathcal{C}_1 = v\underline{g}w + vw$ (see Fig. 4-b). Suppose $m_2^*$ can make use of both $v$ and $w$, say $m_2^*\underline{o_1}v$ and $m_2^*\underline{o_1}w$, to connect nodes on $\mathring{v}\underline{g}\mathring{w}$. We have $r, o_3 \notin V(m_2^*\underline{o_1}v \cup m_2^*\underline{o_1}w)$, since $\mathcal{P}_1$ and $\mathcal{P}_2$ must traverse $r$ to connect $\mathring{v}\underline{g}\mathring{w}$ when $v$ and $w$ are not used. In addition, it is impossible that $m_2^*\underline{o_1}v$ and $m_2^*\underline{o_1}w$ must have an unavoidable common node, say $o_2$, with $m_1^*\underline{e_1}r$; otherwise, $m_1^*$ cannot connect to $g$ when $o_2$ and $r$ are deleted. Thus, $m_2^*\underline{o_1}v$ and $m_2^*\underline{o_1}w$ which do not have common node $o_2$ can be discovered. Then we reselect $\mathcal{C}_2$, i.e., $\mathcal{C}_2 = v\underline{o_1}w + vw$ with $\mathcal{P}_2 = m_2^*\underline{e_2}o_1$ and $\mathcal{P}_1 = m_1^*\underline{e_1 r o_3}g$.

(iv). According to (ii) and (iii), $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{C}_2$ can be discovered to make sure $\mathcal{P}_1$ and $\mathcal{P}_2$ do not have common node $r$. However, if $g$ and $s$ in Fig. 4-b are the same node (see Fig. 4-c), we can also prove it is impossible. In this case, $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \{r\}$ with $m_1, m_2 \notin V(\mathcal{D}_1)$ and $m_1, m_2 \notin V(\mathcal{D}_2)$. For the two cycles, we have $|\mathcal{D}_1| \geq 1$ and $|\mathcal{D}_2| \geq 1$ (since $vw$ is the only common link between $\mathcal{C}_1$ and $\mathcal{C}_2$); therefore, nodes in $\mathcal{D}_1$ ($\mathcal{D}_2$) without monitoring nodes are separated when $r$ and $v$ ($w$) are deleted, contradicting Lemma II.3.

(v). Therefore, $\mathcal{P}_1$ and $\mathcal{P}_2$ without common nodes can be discovered. Accordingly, it is obvious that $\mathcal{P}_1$ and $\mathcal{P}_2$ do not have common links, since a common link means two common nodes (end nodes of this link) between $\mathcal{P}_1$ and $\mathcal{P}_2$. Consequently, $\mathcal{P}_1$ and $\mathcal{P}_2$ with $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ can be discovered.

*6)* If $L(\mathcal{P}) \cap L(\mathcal{C} - v - w) \neq \emptyset$, simply use the first common node as the end node of $\mathcal{P}$.

*7)* We first consider $\mathcal{P}_1$. In $\mathcal{G} - m_2^*$, if $\mathcal{P}_1$ must traverse an end node of $vw$, say $v$, to connect $m_1^*$ and a node on $\mathcal{C}_1 - v - w$, then nodes on $\mathcal{C}_1 - v - w$ are disconnected to $m_1^*$ when $v$ and $m_2^*$ are deleted, contradicting Lemma II.3. Thus, it is impossible that $\mathcal{P}_1$ must traverse an end node of $vw$. However, if $\mathcal{P}_1$ cannot avoid one of $v$ and $w$ to connect $m_1^*$ and $\mathcal{C}_1 - v - w$, then two paths can be constructed. Let $v\underline{e_1 g}w + vw$ be $\mathcal{C}_1$ (see Fig. 4-d). The constructed two paths, connecting $m_2^*$ and $g$, are $m_2^*\underline{e_3 t v e_1}g$ and $m_2^*\underline{e_3 t e_2 w}g$ with $m_2^*\underline{e_3 t v} \cap \mathring{v}\underline{e_1 g}\mathring{w} = \emptyset$ and $m_2^*\underline{e_3 t e_2 w} \cap \mathring{v}\underline{e_1 g}\mathring{w} = \emptyset$ (If they have intersections, $\mathcal{P}_1$ does not have to traverse $v$ and $w$ to connect to a node on $\mathring{v}\underline{e_1 g}\mathring{w}$). Thus, according to Lemma II.3, $g$ must have a connection to $m_1^*$, $m_1^*\underline{e_4}g$, with $m_1^*\underline{e_4}g \cap m_2^*\underline{e_3}t = \emptyset$ (if $m_1^*\underline{e_4}g \cap m_2^*\underline{e_3}t \neq \emptyset$, then $\mathcal{P}_1$ does not have to traverse $v$ and $w$ to connect to a node on $\mathring{v}\underline{e_1 g}\mathring{w}$). Therefore, $\mathcal{C}_2$ can be chosen as $\mathcal{C}_2 = v\underline{t e_2}w + vw$ with $\mathcal{P}_2 = m_2^*\underline{e_3}t$ and $\mathcal{P}_1 = m_1^*\underline{e_4}g$. These two cycles and paths enable $vw$ to be a non-border-link identifiable via the method proposed in Section IV-B1 of [1]. Therefore, non-border-link $vw$ is capable of constructing two cycles and $\mathcal{P}_1$, $\mathcal{P}_2$ with $v, w \notin V(\mathcal{P}_1)$ and $v, w \notin V(\mathcal{P}_2)$. When considering $\mathcal{P}_2$, the same conclusion can be obtained. ■

[8]A vertex which separates two other vertices in the same graph is a *cutvertex*.

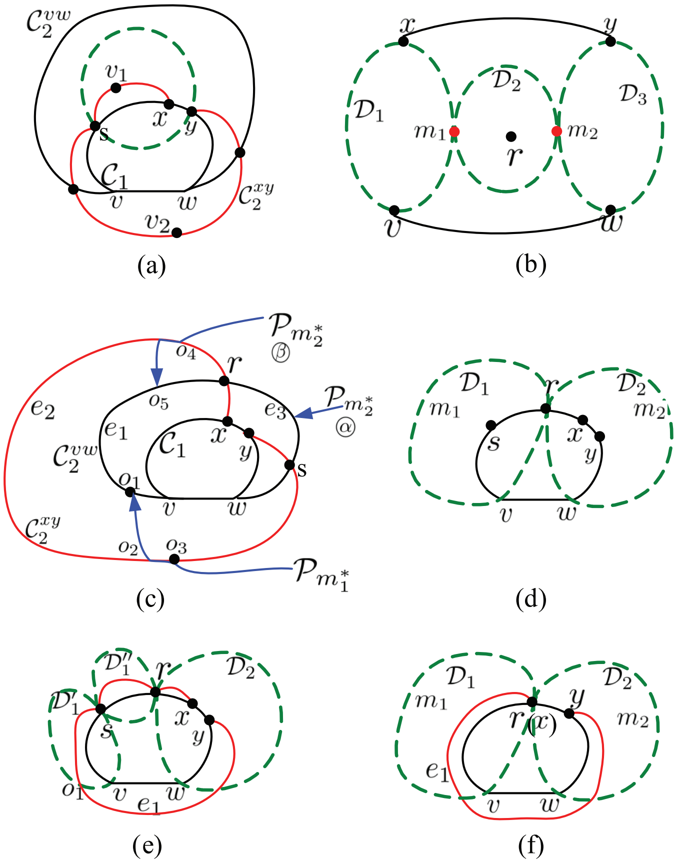


Fig. 5. Border link $vw$ and $xy$ cannot be in the same face.

### *E. Proof of Proposition II.5-(a)*

Using the method to calculate Type 1 identifiable link (Section IV-B1 of [1]), all non-border-links in $\mathcal{H}$ can be identified. While for border-links, they can be categorized into two classes: (i) Class 1. $V(\mathcal{C}_1 \cap \mathcal{C}_2) = \{v, w\}$ and all $\mathcal{P}_1$ must have a common node with $\mathcal{C}_2 - v - w$, and (ii) Class 2. $V(\mathcal{C}_1 \cap \mathcal{C}_2) = \{v, w, r\}$, where $r$ is another unavoidable common node.

Let $vw$ be the border-link in $\mathcal{H}$ and $vw \in L(\mathcal{C}_1)$. All other links on $\mathcal{C}_1$ can use the same face, because $\mathcal{C}_2'$ of other links cannot be disconnected to monitoring nodes when $\mathcal{C}_1$ is deleted.

*1)* Let $vw$ be a border-link of Class 1.

(i). In Fig. 5-a, suppose $xy$ is border-link of Class 2 on $\mathcal{C}_1$, then there is a common node $s$ (there is at most one common node apart from $x$ and $y$, proved in Lemma II.4) on $\mathcal{C}_1$ and $\mathcal{C}_2^{xy}$. Since $\mathcal{C}_1$ is an induced graph, there must be a node, say $v_1$, on $s\underline{v_1}x$ and a node, say $v_2$, on $s\underline{v_2}y$. For all paths connecting $v_1$ and monitoring nodes, they must traverse $s$ or $y$. Therefore, $v_1$ cannot have other connections to $\mathcal{C}_2$ via bypassing $s$ and $y$. Meanwhile, if $v_1$ has a path to one monitoring node in $\mathcal{G} \backslash \mathcal{C}_2^{vw}$, then $x$ has a path to the same monitoring node in $\mathcal{G} \setminus \mathcal{C}_2^{vw}$ as well, contradicting the assumption that $vw$ is a Class 1 border-link. Thus, when $s$ and $y$ are deleted, $v_1$ is separated from $m_1$ and $m_2$, contradicting Lemma II.3. This conclusion also holds when $xy$ and $s$ have common nodes with $vw$. As the position of $s$ alters, however, the separated node might change. For instance, when $s = w$, $v_2$ is separated from $m_1$ and $m_2$ when $x$ and $w$ are deleted. Therefore, $xy$ cannot be a border-link of Class 2 on face $\mathcal{C}_1$.

(ii). Suppose there is another border-link $xy$ of Class 1 and both $\mathcal{C}_2^{vw}$ and $\mathcal{C}_2^{xy}$ must traverse $m_1$ and $m_2$. Then graph $\mathcal{G}$ can be reorganized as Fig. 5-b, which is composed of component $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and link $vw$, $xy$. There is at least one node, say $r$, in $\mathcal{D}_2$, because we have assumed direct link $m_1m_2$ does not exist in $\mathcal{G}$. Thus, the graph is disconnected when $m_1$ and $m_2$ are deleted, contradicting Lemma II.3. Therefore, it is impossible that $\mathcal{C}_2^{vw}$ and $\mathcal{C}_2^{xy}$ must traverse both $m_1$ and $m_2$.

(iii). Since $vw$ is a Class 1 border-link, all possible $\mathcal{P}_1$ must intersect $\mathcal{C}_2^{vw}$. Thus, there exist path $\mathcal{P}_{m_1^*} := \mathcal{P}(m_1^*, v_1)$ and $\mathcal{P}_{m_2^*} := \mathcal{P}(m_2^*, v_2)$ with $v_1, v_2 \in V(\mathcal{C}_2^{vw})$ and $\mathcal{P}_{m_1^*} \cap \mathcal{P}_{m_2^*} = \emptyset$ (If $\mathcal{P}_{m_1^*} \cap \mathcal{P}_{m_2^*} \neq \emptyset$, the common node is a cutvertex). Suppose there is another border-link $xy$ of Class 1 on $\mathcal{C}_1$ (see Fig. 5-c). Then the associated $\mathcal{C}_2^{xy}$ ($V(\mathcal{C}_2^{xy} \cap \mathcal{C}_1) = \{x, y\}$) must have two intersections (since both $vw$ and $xy$ are Class 1 border-link) with $\mathcal{C}_2^{vw}$, say $r$ and $s$ (we have proved that $r$ and $s$ cannot be both monitoring nodes in the previous step). Since $xy$ is another Class 1 border-link, if $\mathcal{P}_{m_1^*}$ connects to $\mathring{r}\underline{e_1vw}\mathring{s}$, it must have intersections with $\mathring{r}\underline{e_2}\mathring{s}$, say the intersection is $o_3$ (the number of intersections maybe greater than one, say both $o_2$ and $o_3$). In addition, we have $o_3 \neq r \neq s$, since if $o_3$ overlaps with $r$ or $s$, then it means $v$ cannot connect to monitoring nodes when $r$ and $s$ are deleted, which is impossible. In Fig. 5-c, let $o_1$ be another node, which can be equal to $v$, on $\mathcal{C}_2^{vw}$. Now we consider the locations of $\mathcal{P}_{m_1^*}$ and $\mathcal{P}_{m_2^*}$. If $\mathcal{P}_{m_2^*}$ ends at $r\underline{e_3}s$ (location @ in Fig. 5-c), then $\mathcal{P}_{m_1^*}$ cannot end at $\mathring{r}\underline{e_1vw}\mathring{s}$, because $xy$ can select $x\underline{re_3s}y + xy$ as $\mathcal{C}_2^{xy}$, and then path $m_1^*\underline{o_3o_2o_1}v$ connecting $m_1^*$ and $v$ does not intersect with the newly selected $\mathcal{C}_2^{xy}$, resulting $xy$ to be a non-border-link, contradicting the assumption that $xy$ is a border-link. Therefore, $\mathcal{P}_{m_1^*}$ also ends at $\mathring{r}\underline{e_3}\mathring{s}$. In this case, however, $v$ is disconnected to monitoring nodes when $r$ and $s$ ($r$ and $s$ cannot be both monitoring nodes) are deleted, contradicting Lemma II.3. Now we change the location of $\mathcal{P}_{m_2^*}$. If no $\mathcal{P}_{m_1^*}$ and $\mathcal{P}_{m_2^*}$ end at $r\underline{e_3}s$, then both $\mathcal{P}_{m_1^*}$ and $\mathcal{P}_{m_2^*}$ (location ß in Fig. 5-c) end at $\mathring{r}\underline{e_1vw}\mathring{s}$. In this case, $\mathcal{C}_2^{xy}$ can be reselected, i.e., $\mathcal{C}_2^{xy} = x\underline{re_3s}y + xy$ with $\mathcal{P}_2^{xy} = m_2^*\underline{o_4}r$ and $\mathcal{P}_1^{xy} = m_1^*\underline{o_3o_2o_1}v$. Thus, $xy$ with $\mathcal{P}_1^{xy} \cap \mathcal{P}_2^{xy} = \emptyset$, a Type 1 identifiable link (Section IV-B1 of [1]), is not a border-link, contradicting the assumption of $xy$ being a border-link. This conclusion also holds when $y = w$ (or $x = v$). Thus, $\mathcal{C}_1^{vw}$ cannot have another border-link of Class 1.

*2)* Let $vw$ be a border-link of Class 2. For $vw$, suppose all cycles must traverse $r$, then $\mathcal{G}$ consists of component $\mathcal{D}_1$ $\mathcal{D}_2$ and link $vw$ (see Fig. 5-d). In addition, each of $\mathcal{D}_1$ and $\mathcal{D}_2$ has a monitoring node in it; otherwise, $\mathcal{D}_1$ ($\mathcal{D}_2$) is separated from monitoring nodes when $r$ and $v$ ($w$) are deleted, contradicting Lemma II.3.

(i). Suppose $xy \in L(\mathcal{D}_2)$ (see Fig. 5-e) is a Class 2 border-link on the same face $\mathcal{C}_1$, all $\mathcal{C}_2^{xy}$ must traverse a node, say $s$,

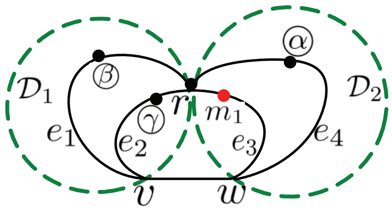


Fig. 6. Border-links and monitoring nodes do not in the same face.

on $\mathcal{C}_1$. If $s$ is on $\mathring{v}\underline{s}\mathring{r}$, $\mathcal{D}_1$ is further split into two components ($\mathcal{D}_1'$ and $\mathcal{D}_1''$), contradicting the claim that $\mathcal{C}_1$ and $\mathcal{C}_2^{xy}$ cannot have two common nodes. Thus, $s$ cannot be on $\mathring{v}\underline{s}\mathring{r}$. If $s = r$ or $s = v$ or $s = w$, then path $o_1\underline{e_1}y$ is required. Since $vw$ is Class 2 border-link, $o_1\underline{e_1}y$ must traverse $r$ as well, resulting $\mathcal{C}_2^{xy} - xy$ containing a cycle, contradicting the basic requirement in [1]. To avoid employing cycles, $\mathcal{C}_2^{xy}$ must be in $\mathcal{D}_2$, in which case nodes, say $s$ (see Fig. 5-d) with $s \in V(\mathcal{C}_1 \cap D_1)$, on $\mathcal{C}_1 - v - w$ has a connection to $m_1$ without intersecting $\mathcal{C}_2^{xy}$. When $s \in V(\mathring{r}\underline{xy}\mathring{w})$ ($s \neq x, s \neq y$) or $x = r$ or $y = w$ or $xy \in L(\mathcal{D}_1)$, the same conclusion can be made. Thus, $xy$ cannot be a Class 2 border-link.

(ii). Suppose $xy \in L(\mathcal{D}_2)$ (see Fig. 5-f) is a Class 1 border-link on the same face $\mathcal{C}_1$, we have $r = x$ or $r = y$, since $\mathcal{C}_1$ and $\mathcal{C}_2^{xy}$ cannot have common nodes, apart from $x$ and $y$. If $r = x$, there should be path $r\underline{e_1}y$ and $r\underline{e_1}y$ cannot have any links outside $\mathcal{D}_1$ and $\mathcal{D}_2$; therefore, $r\underline{e_1}y \subset \mathcal{D}_2$. In this case, there is a path $\mathcal{P}(m_1^*, v)$ ($r \notin V(\mathcal{P}(m_1^*, v))$. If $r$ must be on $\mathcal{P}(m_1^*, v)$, then $v$ is disconnected to monitoring nodes when $r$ and $w$ are deleted.) connecting $m_1^*$ and $v$ without intersecting $r\underline{e_1}y$, contradicting the assumption that $xy$ is a Class 1 border-link. The same conclusion can be obtained when $r = y$. Thus, $xy$ cannot be a Class 1 border-link.

Therefore, a face with a border-link cannot have another border-link. ■

### *F. Proof of Proposition II.5-(b)*

If $vw$ belongs to Class 1, then all paths connecting nodes on $\mathcal{C}_1 - v - w$ and monitoring nodes must intersect $\mathcal{C}_2$. Therefore, $m_1$ and $m_2$ cannot be on $\mathcal{C}_1$. If $vw$ belongs to Class 2 and all paths (besides direct link $vw$) connecting $v$ and $w$ must traverse a monitoring node, say $m_1$, then it means $r = m_1$ in Fig. 5-d. Thus, $m_2$ is in either $\mathcal{D}_1$ or $\mathcal{D}_2$ (each component at least has two links; otherwise, the single link becomes a bridge when $vw$ is deleted). Suppose $m_2$ is in $\mathcal{D}_1$, then $\mathcal{D}_2$ is separated from monitoring nodes when $r$ ($r = m_1$) and $v$ are deleted, contradicting Lemma II.3. Then obviously, it is impossible that $\mathcal{C}_1$ must traverse both $m_1$ and $m_2$. Now suppose either $m_1$ or $m_2$ must be on $\mathcal{C}_1$. Without loss of generality, let $m_1 \in V(\mathcal{D}_2)$ be on $\mathcal{C}_1$ (see Fig. 6). If $m_2$ is at location @, then $\mathcal{D}_1$ is separated from monitoring nodes when $r$ and $w$ are deleted. If $m_2$ is at location β, then $\mathcal{C}_1 = v\underline{e_2}re_4w + vw$ is reselected. If $m_2$ is at location γ, then $\mathcal{C}_1 = v\underline{e_1}re_4w + vw$ is reselected. Therefore, for every border-link $vw \in L(\mathcal{H})$, it can discover a face without traversing $m_1$ and $m_2$. ■

### *G. Proof of Proposition II.5-(c)*

$\mathcal{P}(m_1^*, v)$ and $\mathcal{P}(m_2^*, w)$ exist, since $\mathcal{G}$ is a 2-vertex-connected graph. If $\mathcal{P}(m_1^*, v) \cap \mathcal{P}(m_2^*, w) \neq \emptyset$, let $r \in V(\mathcal{P}(m_1^*, v) \cap \mathcal{P}(m_2^*, w))$, then $v$ and $w$ cannot connect to $m_1^*$ or $m_2^*$ when $r$ is deleted, contradicting Lemma II.3. Based on Proposition II.5-(b), $m_1^*, m_2^* \notin V(\mathcal{C}_1)$. If $\mathcal{P}(m_1^*, v)\mathring{v}$ must have a common node, say $s$, with $\mathcal{C}_1$, then $m_1^*$ cannot connect to $v$ when $s$ is deleted. Thus, $\mathcal{P}(m_1^*, v)$ with $\mathcal{P}(m_1^*, v)\mathring{v} \cap \mathcal{C}_1 = \emptyset$ can be found. Analogously, $\mathcal{P}(m_2^*, w)$ with $\mathcal{P}(m_2^*, w)\mathring{w} \cap \mathcal{C}_1 = \emptyset$ can also be found. ■

## References


[1] L. Ma, T. He, K. K. Leung, A. Swami, and D. Towsley, "Topological constraints on identifying additive link metrics via end-to-end paths measurements," *submitted to Annual Conference of The International Technology Alliance (ACITA)*, 2012.

[2] R. Diestel, *Graph theory*. Springer-Verlag Heidelberg, New York, 2005.