Throughput of proportional fair scheduling over Rayleigh fading channels

E. Liu and K.K. Leung

The proportional fair scheduling (PFS) algorithm is implemented in current 3G wireless networks for high data rate delay-tolerant services. Though the algorithm has low implementation complexity, the problem of proportional fairness is NP-hard. An analytical expression is obtained to approximate the throughput of PFS in cellular networks over Rayleigh fading channels. Comparisons against simulation results show that the expression is accurate.

Introduction: Among various related researches on scheduling, the proportional fair scheduling (PFS) algorithm has been widely conceived as an attractive solution providing a good compromise between the maximum throughput and user fairness in a fading wireless environment. PFS was first introduced by Kelly [1] and is the most commonly cited fair resource allocation. Since its presence [1], the PFS algorithm has aroused considerable interest (see [2–5] and the References therein) and is implemented in 3G wireless networks for high data rate delay-tolerant services [6]. Though the algorithm itself has low implementation complexity, it is known that the problem of proportional fairness is NP-hard [4, 5], and in most researches the results are obtained from computer simulations. The authors of [7] and [2] conducted an asymptotic analysis of the PFS algorithm. However, none of the researches aforementioned provides an analytic expression of the PFS throughput. The authors of [8] and [9] analysed the PFS algorithm with the objective of obtaining an analytic expression for the throughput. The drawback of [8] is that it uses a modified preference metric to simplify the analysis and the result is valid only for a small signal-to-noise ratio (SNR), while in [9] the analytical expression for the PFS throughput in an N-user cellular network requires having the cumulative distribution function (cdf) of the maximum of N fading random variables, which in practice is not known for most cases.

In this Letter, we provide an expression to quantify the PFS throughput in cellular networks over Rayleigh fading channels without the limitations mentioned above. The formula can be efficiently evaluated numerically without resorting to lengthy Monte Carlo simulations, which makes it particularly attractive.

System model: Consider a cellular network where there are N users (n_1, $n_2, \ldots, n_N$) wishing to transmit to the base station (BS), and the rates of transmission are randomly varying. The user with the maximum preference metric will be selected for transmission at the next scheduling slot. The PFS algorithm is described mathematically as follows. Time is divided into small scheduling intervals, called slots. The end of slot m is called time m. In the next time slot $m+1$, the feasible rate of user j is $R_j[m+1]$. Its k-point moving average throughput up to time m is denoted by $\mu_j[m]$, and the preference metric by $M_j[m+1] = R_j[m+1]/\mu_j[m]$. User $i = \arg\max_j M_j[m+1] = \arg\max_j R_j[m+1]/\mu_j[m]$ is scheduled by the BS to transmit in the next time slot $m+1$. The moving average throughput of user j up to time $m+1$ is updated by

$$\mu_j[m+1] = \left(1 - \frac{1}{k}\right)\mu_j[m] + I_j[m+1] \times \frac{R_j[m+1]}{k} \tag{1}$$

where $I_j[m+1]$ is the indicator function of the event that user j is scheduled to transmit in time slot $m+1$:

$$I_j[m+1] = \begin{cases} 1, & j \text{ is scheduled in slot } m+1 \\ 0, & \text{else} \end{cases} \tag{2}$$

Assuming first-order wide-sense stationary $\mu_{j;k}$ and applying Bayes's theorem, we have

$$\begin{aligned} E[\mu_j[m]] &= E[R_j[m+1] | I_j[m+1] = 1] = Pr(I_j[m+1] = 1) \\ &= Pr(I_j[m+1] = 1) \int_0^{\infty} x f_{R_j}(x | I_j[m+1] = 1)\, dx \\ &= \int_0^{\infty} x f_{R_j}(x) Pr(I_j[m+1] = 1 | R_j[m+1] = x)\, dx \end{aligned} \tag{3}$$

where $E[\cdot]$ denotes the statistical average, $Pr(I_j[m+1] = 1)$ is the average probability that node j will be scheduled in slot $m+1$, $Pr(I_j[m+1] = 1 | R_j[m+1] = x)$ is the corresponding conditional probability and $f_{R_j}(\cdot)$ is the probability density function (pdf) of R_j.

We would like to clarify that the moving average throughput μ is the measured average throughput over k slots and is the 'instantaneous' throughput commonly seen in the literature, while $E[\cdot]$ is the average throughput over ∞ slots.

Equation (1) indicates that μ_j varies slowly with R_j for large k, making random variable $R_j[m+1]/\mu_j[m]$ a good approximation of $R_j[m+1]/E[\mu_j]$. Experiments suggest that this approximation is valid when $k \geq 50$ and $N \geq 5$, with an accuracy greater than 98%. We thus have

$$\begin{aligned} &Pr(I_j[m+1] = 1 | R_j[m+1] = x) \\ &\qquad \simeq Pr\left(\forall i \neq j, \frac{R_i[m+1]}{E[\mu_i]} < \frac{x}{E[\mu_j]}\right) \end{aligned} \tag{4}$$

We can then rewrite (3) as

$$E[\mu_j] = \int_0^{\infty} x f_{R_j}(x) \prod_{\forall i \neq j, i=1}^{N} F_{R_i}(E[\mu_i] \times x / E[\mu_j])\, dx \tag{5}$$

where $F_{R_i}(\cdot)$ is the cdf of R_i.

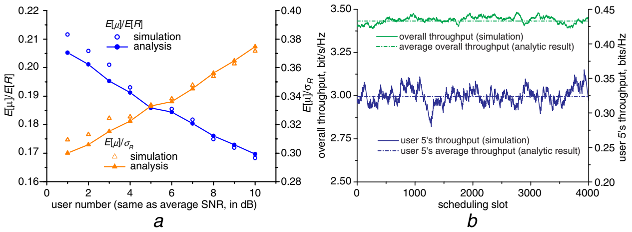

Fig. 1 *PFS over Rayleigh fading channels: analysis against simulation*

Simulations (see Fig. 1*a*) have found that, for PFS over Rayleigh fading channels, if $E[R_i] \geq E[R_j]$,

$$\frac{\sigma_{R_i}}{\sigma_{R_j}} \leq \frac{E[\mu_i]}{E[\mu_j]} \leq \frac{E[R_i]}{E[R_j]} \tag{6}$$

where σ_{R_i} is the standard deviation of R_i.

Denote $E[R_j]/\sigma_{R_j}$ by M_j and after long mathematical manipulations together with (6), we have the upper and lower bounds of $E[\mu_j]$:

$$\begin{aligned} E[\mu_j] \leq \sigma_{R_j} &\int_{-M_j}^{\infty} (y\sigma_{R_i} + E[R_j]) f_{R_j}(y\sigma_{R_j} + E[R_j]) \\ &\times \prod_{\forall i \neq j, E[R_i] \geq E[R_j]}^{N} F_{R_i}\left(y \frac{E[R_i]}{E[R_j]} \sigma_{R_j} + E[R_i]\right) \\ &\times \prod_{\forall i \neq j, E[R_i] < E[R_j]}^{N} F_{R_i}\left(y\sigma_{R_j} + \frac{\sigma_{R_i}}{\sigma_{R_j}} E[R_j]\right) dy \end{aligned} \tag{7}$$

$$\begin{aligned} E[\mu_j] \geq \sigma_{R_j} &\int_{-M_j}^{\infty} (y\sigma_{R_j} + E[R_j]) f_{R_j}(y\sigma_{R_j} + E[R_j]) \\ &\times \prod_{\forall i \neq j, E[R_i] < E[R_j]}^{N} F_{R_i}\left(y \frac{E[R_i]}{E[R_j]} \sigma_{R_j} + E[R_i]\right) \\ &\times \prod_{\forall i \neq j, E[R_i] \geq E[R_j]}^{N} F_{R_i}\left(y\sigma_{R_i} + \frac{\sigma_{R_i}}{\sigma_{R_j}} E[R_j]\right) dy \end{aligned} \tag{8}$$

It is easy to prove that the following expression lies between the upper and lower bounds given in (7) and (8):

$$\begin{aligned} \sigma_{R_j} &\int_{-M_j}^{\infty} (y\sigma_{R_j} + E[R_j]) f_{R_j}(y\sigma_{R_j} + E[R_j]) \\ &\times \prod_{\forall i \neq j, i=1}^{N} F_{R_i}(y\sigma_{R_i} + E[R_i]) dy \end{aligned} \tag{9}$$

We can then use (9) to approximate $E[\mu_j]$. It has been pointed out that the feasible rate of a link over Rayleigh fading channels can be modelled by a Gaussian distribution with surprisingly high accuracy [10]. For a Rayleigh fading SISO channel, the received SNR at a node is an exponential distribution; we then have

$$E[R] = \int_0^{\infty} \log_2 (1 + \overline{SNR} \times x) \times e^{-x} dx \quad (10)$$

$$\sigma_R^2 \int_0^{\infty} (\log_2 (1 + \overline{SNR} \times x))^2 \times e^{-x} dx - (E[R])^2 \quad (11)$$

where $\overline{SNR}$ is the average received SNR.

Note that, for Gaussian R_i, $f_{R_i}(x) = \rho((x - E[R_i])/\sigma_{R_i})/\sigma_{R_i}$ and $F_{R_i}(x) = \phi((x - E[R_i])/\sigma_{R_i})$ where $\rho(\cdot)$ and $\phi(\cdot)$ are the pdf and cdf of zero mean, unit variance standard normal distribution, respectively. Finally we have

$$\begin{aligned} E[\mu_j] \simeq & \frac{(1 - [\phi(-M_j)]^N)}{N} \times \int_0^{\infty} e^{-x} \times \log_2(1 + \overline{SNR}_i \times x) dx \\ & + \sqrt{\begin{pmatrix} \int_0^{\infty} e^{-x} \times (\log_2(1 + \overline{SNR}_i \times x))^2 dx \\ -\left(\int_0^{\infty} e^{-x} \times \log_2 (1 + \overline{SNR}_i \times x) dx\right)^2 \end{pmatrix}} \\ & \times \int_{-M_j}^{\infty} y\rho(y) \times [\phi(y)]^{N-1} dy \end{aligned} \quad (12)$$

Numerical results: We assume: 4000 scheduling slots, moving average factor $k = 500$, a cellular network of ten users with $\overline{SNR} = 1, 2, \ldots, 10$ dB, where users experience independent Rayleigh fading. For simplicity of presentation, each user is numbered by its $\overline{SNR}$. The simulation and numerical results shown in Fig. 1*a* corroborate (6). Fig. 1*b* shows the throughput of the network and the throughput of user 5. The results validate that simulation and theoretical values are well matched. This is a clear indication that the developed model is suitable for understanding the performance of PFS in a Rayleigh fading environment.

Conclusions: We have developed a model to analyse the throughput performance of PFS in cellular networks. The expression can be easily evaluated using numerical integration. Comparisons with simulation results indicate that the model is accurate in estimating the average throughput of PFS over Rayleigh fading channels.

3 April 2009
doi: 10.1049/el.2009.0943

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