Fair Resource Allocation under Rayleigh and/or Rician Fading Environments

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***Abstract*—Proportional fair scheduling (*PFS*) provides good balance between throughput and fairness via multi-user diversity and game-theoretic equilibrium. Very little analytical work exists on understanding the performance of *PFS*. Most existing prior results are for networks with Rayleigh fading. In this paper, we provide theoretical results for *PFS* in general fading environments. The results reveal that the average throughput of a user solely depends on its own channel statistics when its instantaneous data rate is *Gaussian*. Based on the theoretical results, we analyze the *PFS* performance under various scenarios with *Rayleigh* and/or *Rician* fading, and the numerical results match very well with the simulation ones. To the best of our knowledge, this work is the first one theoretically investigating the *PFS* problem in general fading environments.**

Keywords- proportional fair scheduling, multi-user diversity, Gaussian, Rayleigh fading, Rician fading

I. INTRODUCTION

In the considerable efforts to deal with the tradeoff between optimal system throughput and user fairness, the proportional fair scheduling (*PFS*) algorithm is proposed. Since its presence [1], there has been substantial interest in the *PFS* algorithm in wireless networks (see [2, 3] and the references therein).

Though the *PFS* algorithm has got so much attention and currently implemented in 3G wireless network for high data rate delay-tolerant services [4], one can only see very limited analysis results on *PFS* [5-11], whose derivations typically assume a simple *linear rate model* or *logarithm rate model* and/or use a simplified form of the original *PFS* preference metric. The linear rate model is only valid for networks where the signal to interference-plus-noise ratio (*SINR*) is very small [5], while the logarithm rate model can only be used for single-input-single-output (*SISO*) communications [8]. The use of *linear rate model* or *logarithm rate model* is a reasonable modeling convention. However, when examining throughput performance, it does not seem satisfactory to assume such simplified rate models. Moreover, all existing researches are assuming some kind of independent identically distributed (*i.i.d*) relationship among users in their derivations [5-9, 11]. Undoubtedly, these assumptions limit the use of existing theoretical results on *PFS*.

Our previous work [12] provides an analytic expression for the throughput of *PFS* in such a scenario where the instantaneous data rate over a *Rayleigh* fading channel can be approximated by a *Gaussian* distribution [13, 14]. Different from the *linear rate model* and *logarithm rate model*, the Gaussian approximation (*GA*) method used in [12] applies to both low and high SINR cases, moreover, it can be used to accurately model the instantaneous data rate of a multiple-input-multiple-output (*MIMO*) link over a *Rayleigh/Rician* fading channel. The main drawback of our prior work is that it assumes a proportional relationship between the mean and standard deviation of instantaneous data rate in the derivation.

While most existing results are for *Rayleigh* fading networks, we would like to point out that a realistic network typically has fundamentally different fading processes in practice. This paper extends our prior work to such general fading environments. Specifically, we study the *PFS* problem in 4 different scenarios: *Gaussian* rate network, *Rayleigh* fading network, *Rician* fading, and *Rayleigh*+*Rician* hybrid fading network without the mathematic simplifications above.

The rest of the paper is structured as follows. Related work is discussed in Section II. After that, analytical results which extend our prior work on *PFS* are presented in Section III. Finally, in Section IV, we present simulation results to validate the theoretical findings under various fading environments, followed by the conclusion in Section V.

II. RELATED WORK

Since its first presence in Kelly's seminal paper [1], significant efforts has been put into the study of the PFS algorithm. Holtzman [5] conducted the asymptotic analysis of the *PFS* algorithm, with a result that all other things being equal, the user class with more fading variability gets more throughput. Kushner et al. [7] investigated the convergence property of the algorithm. They stated that the limiting behavior of the throughputs converges to the solution of an ordinary differential equation, and found the limit throughput is proportional to the average instantaneous rate for *Rayleigh* fading by assuming the instantaneous rate is proportional to the received *SINR* which is *i.i.d* for all users. Also, [11] presented some results on *PFS* for the scenario where the relative rate fluctuations are statistically identical, stating that each user would receive same amount of the time slots.

To simplify the problem, most existing analytic results are assuming some kind of *i.i.d* relationship among users [5-9, 11]. For example, [6, 7] assume the *SINR* of each user is an *i.i.d* *Exponential* distribution; [8] takes the assumption similar to [6] that user *k's SINR* $S_k=c_k\times C$ ($\forall k$), where c_k is a user-related constant and C is an *i.i.d Exponential* distribution random variable independent of all users, and [11] assumes that the instantaneous data rate of user i with time-average rate C_i is distributed as $R_i=C_iY_iZ$, where Y, Y_1, Y_2, … are *i.i.d* copies and Z represent a possible correlation component with unit mean, and the exponentially smoothed throughput of user i scale linearly with the time-average rate C_i, i.e., $W_i=C_iV_i$, where the random variables $V_1,\dots, V_m$ are identically distributed.

On the other hand, *linear rate model* and *logarithm rate model* are the two rate models commonly used for analyzing the performance of *PFS*. For example, [5, 6, 11] use the *linear rate model*, while [8] uses the *logarithm rate model*. In the *linear rate model*, the instantaneous data rate R of a user is linearly proportional to the received *SINR*, i.e, $R_i = \beta \times SINR_i$ for any user i; while in the *logarithm rate model*, there is a logarithmic relationship between the received *SINR* and the instantaneous data rate, i.e., $R_i \propto Log_2(1+\beta \times SINR_i)$. These two rate models have their merits as they greatly simplify the mathematical analysis of *PFS*. However, such simplified models may not be satisfactory enough to characterize the instantaneous data rate in practical network environments, especially where there are fundamentally different fading processes. As stated in [5], the *linear rate model* is usually a reasonable approximation for small *SINR* and is not accurate when multiple modulations or codings are used. On the other hand, the *logarithm rate model* can only be used for *SISO* links and is a very rough approximation when used for *MIMO* links [8]. In fact, works by Telatar [13] and Smith et al. [14] suggested that the instantaneous rate over *Rayleigh* or *Rician* fading channels can be better modeled by a *Gaussian* distribution with surprisingly high accuracy. Thus the *Gaussian* approximation (*GA*) instead of the two simplified rate models is used in our prior research to study the *PFS* performance [12]. The main drawback of [12] is that it assumes a proportional relationship between the mean and standard deviation of instantaneous data rate.

It is not surprising that most studies mentioned above are for *Rayleigh* fading networks as a system with more fading variability will get more *PFS* benefits [5]. In fact, the channel characteristics in a realistic network should be better modeled as a mixture of *Rayleigh* and *Rician* fading. This paper will also consider such fading scenarios.

In all, our goal is to study the *PFS* performance in general fading wireless networks without the mathematic simplifications used in existing researches.

III. PFS: Theoretical Results

We first outline our previous results. Then we present the analytical expression of *PFS* in general fading environments.

In [12], we have the following important result:

In an N-user cellular network implementing the PFS algorithm, the average throughput of user j satisfies

$$E[\mu_j] = \int_0^\infty x f_{R_j}(x) \Pr\left(I_j = 1 \middle| R_j = x\right) dx \tag{1}$$

where $\Pr(I_j=1|R_j=x)$ *is the conditional probability of user j to be scheduled given its instantaneous data rate* R_j,

$$\Pr\left(I_j = 1 \middle| R_j = x\right) = \prod_{i=1, i\neq j}^{N} F_{R_i}\left(\frac{E[\mu_i]}{E[\mu_j]} x\right) \tag{2}$$

$f_{Ri}(.)$, $F_{Ri}(.)$ *are the probability density function (pdf) and cumulative distribution function (cdf) of the instantaneous data rate of user i,* μ_i *is the throughput of user i,* $E[.]$ *is the statistical average, and* I_j *is the indicator function of the event that user j is scheduled to transmit in next time slot,*

$$I_j = \begin{cases} 1, & j \text{ is scheduled to transmit in next time slot} \\ 0, & else \end{cases} \tag{3}$$

Simulations (Figure 4 and 5) have found that, in *Rayleigh*, *Rician* or *Rayleigh+Rician* hybrid fading environments, $E[\mu]/E[R]$ is a monotonically decreasing function of $E[R]$, while $E[\mu]/\sigma_R$ is a monotonically increasing function of σ_R, i.e., $\sigma_{Ri}/\sigma_{Rj} \le E[\mu_i]/E[\mu_j] \le E[R_i]/E[R_j]$ given $0 \le E[R_j] \le E[R_i]$. Denote $E[R_j]/\sigma_{Rj}$ by M_j, we have

$$\begin{aligned} E[\mu_j] \le \sigma_{R_j} \int_{-M_j}^{\infty} & \left(y\sigma_{R_j} + E[R_j]\right) f_{R_j}\left(y\sigma_{R_j} + E[R_j]\right) \\ & \times \left[\prod_{\forall i\neq j, E[R_i]\ge E[R_j]}^{N} F_{R_i}\left(y \frac{E[R_i]}{E[R_j]} \sigma_{R_j} + E[R_i]\right)\right] \\ & \times \left[\prod_{\forall i\neq j, E[R_i] < E[R_j]}^{N} F_{R_i}\left(y\sigma_{R_i} + \frac{\sigma_{R_i}}{\sigma_{R_j}} E[R_j]\right)\right] dy \end{aligned} \tag{4}$$

and

$$\begin{aligned} E[\mu_j] \ge \sigma_{R_j} \int_{-M_j}^{\infty} & \left(y\sigma_{R_j} + E[R_j]\right) f_{R_j}\left(y\sigma_{R_j} + E[R_j]\right) \\ & \times \left[\prod_{\forall i\neq j, E[R_i]\ge E[R_j]}^{N} F_{R_i}\left(y\sigma_{R_i} + \frac{\sigma_{R_i}}{\sigma_{R_j}} E[R_j]\right)\right] \\ & \times \left[\prod_{\forall i\neq j, E[R_i] < E[R_j]}^{N} F_{R_i}\left(y \frac{E[R_i]}{E[R_j]} \sigma_{R_j} + E[R_i]\right)\right] dy \end{aligned} \tag{5}$$

If we write

$$\begin{aligned} E[\mu_j] \approx \sigma_{R_j} \int_{-M_j}^{\infty} & \left(y\sigma_{R_j} + E[R_j]\right) f_{R_j}\left(y\sigma_{R_j} + E[R_j]\right) \\ & \times \prod_{\forall i=1, i\neq j}^{N} F_{R_i}\left(y\sigma_{R_i} + E[R_i]\right) dy \end{aligned} \tag{6}$$

It is easy to prove that the average throughput given by (6) lies between the upper and lower bounds given by (4) and (5), respectively.

Now we have the closed-form expressions (4)-(6) for the *PFS* throughput.

It is easy to prove that with *GA* method for *Rayleigh* and/or *Rician* fading environments, (6) further reduces to

$$\begin{aligned} E[\mu_j] \approx & \frac{E[R_j]}{N} \times \left(1 - \left[F_{(0,1)}\left(-E[R_j]/\sigma_{R_j}\right)\right]^N\right) \\ & + \int_{-M_j}^{\infty} y\sigma_{R_j} f_{(0,1)}(y) \times \left[F_{(0,1)}(y)\right]^{N-1} dy \end{aligned} \tag{7}$$

Moreover, if user j's instantaneous data rate $R_j=c_j \times SINR$ ($\forall j$), where c_j is a user-related, positive constant and *SINR* is an *i.i.d* exponentially distributed random variable independent of all users, (6) can be written as.

$$\begin{aligned} E[\mu_j] &= \int_0^\infty c_j y \times f_C(y) \times (F_C(y))^{N-1} dy \\ &= \int_0^\infty c_j y \times e^{-y} \times (1-e^{-y})^{N-1} dy = c_j \times \frac{1}{N} \times \sum_{k=1}^{N} \frac{1}{k} \end{aligned} \quad (8)$$

In fact, (8) is the analytical result given by [6] and [8] for a *Rayleigh* fading network using the linear rate model.

Formula (7) has the great practical and theoretical interests of being mathematically graceful and simple.

(7) reveals a very interesting merit of *PFS*: node *j*'s throughput is solely (approximately, if we ignore the effect of M_j on the integral) determined by its own channel statistics when the instantaneous data rate is *Gaussian*.

Other observations can also be made from (7): Firstly, as $E[R_j]/N$ is the mean throughput of node *j* when using round robin (*RR*) scheduling, the second item in the right hand of (7) is in fact the improvement of node *j*'s throughput when using *PFS* instead of *RR* scheduling. Secondly, *PFS* benefits more in severe fading networks where σ_{Rj} is large. On the other hand, *PFS* can be viewed as a *RR* scheduling algorithm in low-fading environments where σ_{Rj} is relatively small compared with $E[R_j]$. Similar observations have been made by others [5, 6, 11].

We now conduct simulations for various fading scenarios to validate our theoretical results.

IV. NUMERICAL RESULTS AND SIMULATION

Throughout, we assume that users have independent fading channels.

We run the simulations for the following scenarios:

- I the instantaneous data rate is *Gaussian*
- II *Rayleigh* fading channel, *MIMO* link;
- III *Rayleigh* fading channel, *SISO* link;
- IV *Rician* fading channel, *SISO* link, and
- V *Rayleigh+Rician* hybrid fading channel, *SISO* link.

Simulation parameters are: 4000 scheduling slots, moving average factor *k*=500. For scenarios other than case I, the instantaneous data rate (in bps/Hz) is given by $\text{Log}_2(1+SINR)$, where *SINR* is a random variable characterized by corresponding fading process.

Case I: Instantaneous data rate is *Gaussian*. Network size varies from 5 to 180 users. For each user, the instantaneous data rate *R* is a *Gaussian* distribution with a mean $E[R]$ uniformly distributed between 1~8bps/Hz[*1], and a standard deviation σ_R uniform distributed between $0\sim0.4E[R]$[*2].

Case II: *Rayleigh* fading network of 10 users with average *SINR*=1,2,3,4,5,6,7,8,9,10 *dB*, respectively. 5×5 *MIMO* is used for the communication between a user and the access point.

Case III: *Rayleigh* fading network of 10 users with average *SINR*=1,2,3,4,5,6,7,8,9,10 *dB*, respectively.

Case IV: *Rician* fading: 10 users with average *SINR*=11,12,13,14,15,16,17,18,19,20 *dB*, respectively.

Case V: *Rayleigh+Rician* hybrid fading: 10 *Rayleigh* fading users with average *SINR*=1,2,3,4,5,6,7,8,9,10 *dB*, respectively, plus 10 *Rician* fading users with average *SINR*=11,12,13,14,15,16,17,18,19,20 *dB*[*3].

For cases III-V, *SISO* is used for the communication between a user and the access point (i.e., base station in cellular networks).

For the simplicity of presentation, each node is numbered by its average *SINR* in cases III~V.

The *SINR* of a *Rayleigh* fading user is an exponential distribution. The mean and variance of *R* (in bps/Hz) over a *Rayleigh* fading channel are then given by

$$E[R] = \int_0^\infty \log_2\left(1+\overline{SINR}\times\lambda\right)\times e^{-\lambda} d\lambda \quad (9)$$

$$\sigma_R^2 = \int_0^\infty \left(\log_2\left(1+\overline{SINR}\times\lambda\right)\right)^2 \times e^{-\lambda} d\lambda - \left(E[R]\right)^2 \quad (10)$$

where $\overline{SINR}$ is the average received *SINR*.

For a *t*×*r MIMO* link over *Rayleigh* fading channel, where *t* and *r* are the numbers of transmit and receive antennas, respectively, there are $m=\min(t,r)$ nonnegative singular values of the channel matrix, each with the same *pdf* [13]

$$p(\lambda) = \frac{1}{m}\sum_{i=0}^{m-1} \frac{i!\,\lambda^{n-m} e^{-\lambda}}{(i+n-m)!}[L_i^{n-m}(\lambda)]^2 \quad (11)$$

where $n=\max(t,r)$, $L_k^{n-m}(.)$ is generalized *Laguerre* polynomials of order *k*. We also assume that the channel information is not known at a sender side, i.e., sender allocates its power equally to all the transmit antennas. In such cases, the mean and variance of a *MIMO* link over a *Rayleigh* fading channel are given by [13] [14]

$$E[R] = \int_0^\infty \omega(\lambda) \sum_{k=0}^{m-1} \frac{k!\,\lambda^{n-m} e^{-\lambda}}{(k+n-m)!}[L_k^{n-m}(\lambda)]^2 d\lambda \quad (12)$$

$$\sigma_R^2 = m\int_0^\infty \omega^2(\lambda)p(\lambda)d\lambda - \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\left[\frac{i!\,j!}{(i+n-m)!(j+n-m)!}\right] \times \left(\int_0^\infty \lambda^{n-m} e^{-\lambda} L_i^{n-m}(\lambda) L_j^{n-m}(\lambda)\omega(\lambda)d\lambda\right)^2 \quad (13)$$

where $\omega(\lambda) = \log_2(1+\overline{SINR}\times\lambda/t)$, $\overline{SINR}$ is the average *SINR*.

[*1] A data rate within 1~8bps/Hz corresponds to a practical modulation type (*BPSK*~256*QAM*).

[*2] Assumed Gaussian, *R* could be negative. So we set $\sigma_R \le 0.4E[R]$ to ensure the probability $Pr\{R<0\}\approx0.6\%$.

[*3] The *SINR* in a *Rician* fading area is typically larger than that in a *Rayleigh* fading area since there is a dominate component in *Rician* fading case.

In a *Rician* fading environment, the *SINR* of a user is a *noncentral chi-square* distribution[*1] [15] with two degrees of freedom and noncentrality parameter ν^2, where ν is the ratio of signal strength in dominant component over the scattered one, so the mean and variance of R over a *Rician* fading channel are given by

$$\begin{cases} E[R]=\int_0^\infty \log_2\left(1+\dfrac{\overline{SINR}\times\lambda}{2+\nu^2}\right)\times\dfrac{e^{-0.5(\lambda+\nu^2)}}{2}I_0(\nu\sqrt{x})d\lambda \\ \sigma_R^2=\int_0^\infty\left(\log_2\left(1+\dfrac{\overline{SINR}\times\lambda}{2+\nu^2}\right)\right)^2\times\dfrac{e^{-0.5(\lambda+\nu^2)}}{2}I_0(\nu\sqrt{x})d\lambda-\left(E[R]\right)^2 \end{cases} \quad (14)$$

where $I_0(.)$ is the modified *Bessel* function of the first kind.

In the simulation, we set noncentrality parameter $\nu^2=25$ for case IV.

The validities of our theoretical results are illustrated in Figure 1~5. In these experiments, we make several important observations:

(1). the *PFS* algorithm is bandwidth-efficient. Figure 1 shows that it achieves a spectral efficiency of about 6~7bps/Hz in networks where the highest possible data rate for a user is 8bps/Hz;

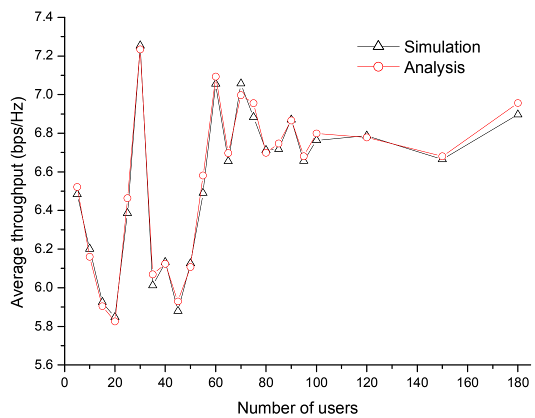

Figure 1. Average overall throughput vs user number: Case I (Gaussian instantaneous data rate with random mean and variance)

(2). Figure 2-5 suggest, while most existing prior results are for *Rayleigh* fading case, our analytical results apply to *Rayleigh*, *Rician* and hybrid fading scenarios;

(3). While existing results only apply to low *SINR* region (*linear rate model*) or *SISO* cases (*logarithm rate model*), Figure 2-5 show that our analysis apply to both *SISO* and *MIMO* networks at low and/or high *SINR*;

(4). Not surprisingly, Figure 3 indicates a higher average throughput for a *Rician* fading user as it has dominant component which results in higher *SINR* than a *Rayleigh* fading user. Nevertheless, it appears from Figure 4 that the throughput gain of *PFS* over *RR* scheduling (i.e., $N\times E[\mu]/E[R]$, where N is the number of users in the network) under a *Rayleigh* fading channel is typically higher than that under a *Rician* fading channel. From Figure 3, it is fair to predict that users with more fading will get more benefits with *SINR* being equal.

(5). Figure 4 and 5 show that, for the *PFS* algorithm in all three fading cases, $E[\mu]/E[R]$ is a monotonically decreasing function of $E[R]$, while $E[\mu]/\sigma_R$ is a monotonically increasing function of σ_R. (From (9)-(14), it is easy to prove that σ_R is a monotonically increasing function of $E[R]$ in *Rayleigh/Rician* fading environments);

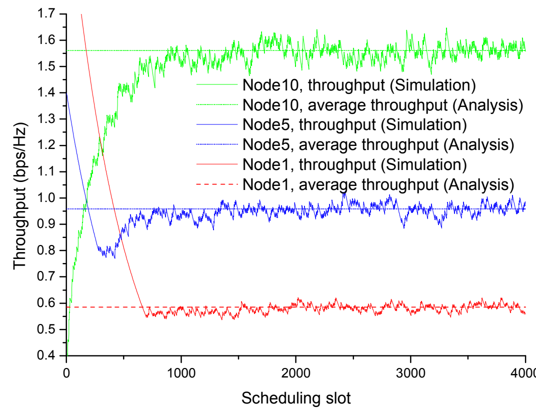

Figure 2. Average overall throughput vs user number Case II (5×5 MIMO, *Rayleigh* fading)

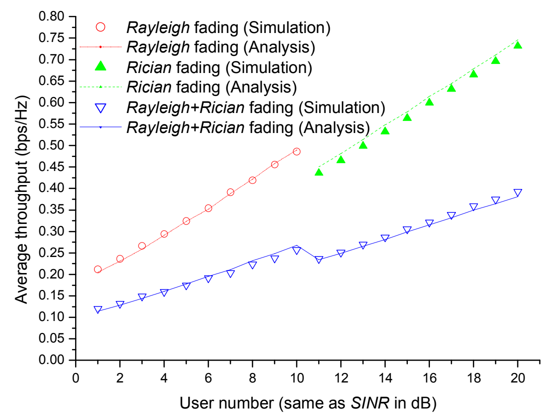

Figure 3. Average throughput of each user Cases III, IV and V (*Rayleigh*, *Rician* and Hybrid fading)

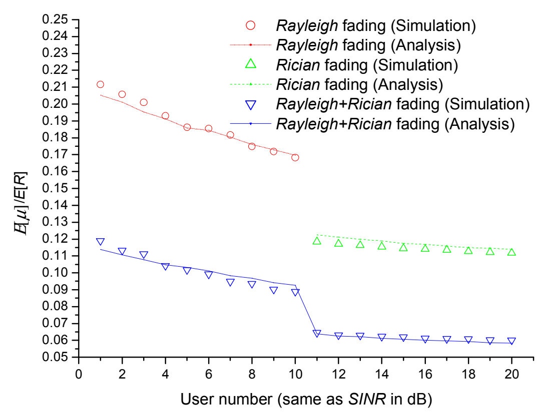

Figure 4. $E[\mu]/E[R]$: monotonically decreasing with respect to $E[R]$: Cases III, IV and V (*Rayleigh*, *Rician* and Hybrid fading)

[*1] The mean of a *noncentral chi-square* distribution with 2 degree of freedom and noncentrality parameter ν^2 is $2+\nu^2$.

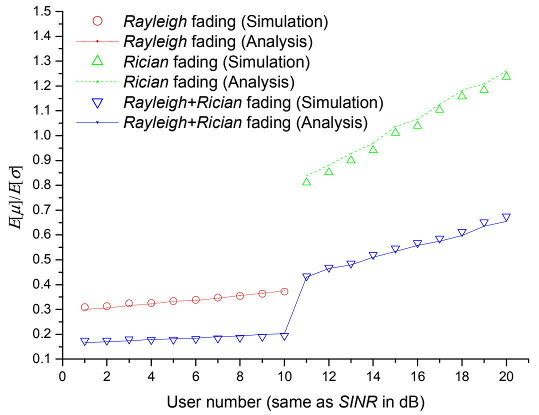

Figure 5. $E[\mu]/\sigma_R$: monotonically increasing with respect to σ_R Cases III, IV and V (*Rayleigh*, *Rician* and Hybrid fading)

The overall conclusion from these experiments is that the closed-form expressions presented in Section III can be used with confidence to evaluate the performance of the *PFS* algorithm under general fading environments.

V. CONCLUSIONS

We have derived a theoretical framework to facilitate researches on *PFS*. Our contributions are summarized as follows.

- We introduce a new framework to analyze *PFS*. It has the great practical and theoretical interests of being mathematically graceful and simple. The analytic framework provides very clear physical meaning of the *PFS* algorithm.
- New exact expressions of the *PFS* throughput are developed. Simulations show that our formulas remain valid for general (i.e., *Rayleigh*, *Rician* and hybrid) fading scenarios. This is in contrast to existing researches that assume *Rayleigh* fading.
- While existing analytic work are developed for *SISO* case, the proposed analytic framework provides an extension to analyze *PFS* in *MIMO* network.
- Our contribution generalizes existing analytical results by removing the *i.i.d* restriction used in literature.

Our analysis provides guideline and theoretical support on system design, simulation-based modeling and performance analysis of the *PFS* algorithm in the context of cross-layer design. The results are being integrated into the MEMBRANE project [16]. For the analysis to be more applicable, we are considering the extension to wireless mesh networks and also its extension to *Nakagami* fading as a general fading distribution.

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REFERENCES

[1] F. Kelly, "Charging and Rate Control for Elastic Traffic", *Eur. Trans. on Telecommun.*, February 1997, pp. 33-37.

[2] T. D. Nguye and Y. Han, "A Proportional Fairness Algorithm with QoS Provision in Downlink OFDMA Systems," *IEEE Commun Lett.,* vol. 10, no. 11, November 2006.

[3] Z. Han, Z. Ji and K. J. Ray Liu, "Fair Multiuser Channel Allocation for OFDMA Networks Using Nash Bargaining Solutions and Coalitions," *IEEE Trans. Commun.,* vol. 53, no. 8, pp. 1366-1376, August 2005.

[4] A. Jalali, R. Padovani and R. Pankai, "Data Throughput of CDMA-HDR: a High Efficiency-High Data Rate Personal Communication Wireless System," in *Proc. Veh. Technol. Conf*., January. 2001, pp. 55-67.

[5] J. M. Holtzman, "Asymptotic analysis of proportional fair algorithm," in *Proc. IEEE PIMRC,* San Diego, CA, 2001, pp. 33-37.

[6] D. Avidor, S. Mukherjee, J. Ling and C. Papadias, "On Some Properties of the Proportional Fair Scheduling Policy," in *Proc. IEEE PIMRC,* New Jersey, September 2004, pp. 853-858.

[7] H. J. Kushner and P. A. Whiting, "Asymptotic Properties of Proportional-Fair Sharing Algorithms: Extensions of the Algorithm," in *Proc. of the Annual Allerton Conference on Communication, Control and Computing,* vol. 41, 2003, pp. 303-311.

[8] J-G. Choi and S. Bahk, "Cell-Throughput Analysis of the Proportional Fair Scheduler in the Single-Cell Environment," *IEEE Trans. Veh. Technol.,* vol. 56, no. 2, pp. 766-778, March 2007.

[9] G. Caire, R. Muller and R. Knopp, "Hard Fairness Versus Proportional Fairness in Wireless Communications: The Single-Cell Case," *IEEE Trans. Inf. Theory.,* vol. 53, no. 4, pp. 1366-1385, April 2007.

[10] M. Andrews, "Instability of the Proportional Fair Scheduling Algorithm for HDR," *IEEE Transactions on Wireless Communications*, Vol.3, No.5, September 2004.

[11] S. Bost, "User-level performance of channel-aware scheduling algorithms in wireless data networks," in *Proc. INFOCOM,* San Francisco, 2003, pp. 321-331.

[12] E. Liu and Kin K. Leung, "Proportional Fair Scheduling: Analytical Insight under Rayleigh Fading Environment," in *Proc. IEEE WCNC,* Las Vegas, March/April 2008.

[13] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Transactions on Telecommunications*, Vol.10, No.6, pp.585-595, November/December 1999.

[14] P. J. Smith and M. Shafi, "On a Gaussian Approximation to the Capacity of Wireless MIMO Systems," in *Proc. IEEE ICC,* New York, April 2002, pp. 406-410.

[15] A. Stuart and J. K. Ord, *Kendall's Advanced Theory of Statistics, Vol. 2A: Classical Inference & the Linear Model, 6th ed.* New York: Oxford University Press, p. 865, 1999.

[16] The MEMBRANE website. [Online]. Available: http://www.imperial.ac.uk/MEMBRANE