LTI Discrete-Time Systems in Transform Domain

Simple Filters
Comb Filters (Optional reading)
Allpass Transfer Functions
Minimum/Maximum Phase Transfer Functions
Complementary Filters (Optional reading)
Digital Two-Pairs (Optional reading)

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Simple Digital Filters

- Later in the course we shall review various methods of designing frequency-selective filters satisfying prescribed specifications
- We now describe several low-order FIR and IIR digital filters with reasonable selective frequency responses that often are satisfactory in a number of applications

Simple FIR Digital Filters

- FIR digital filters considered here have integer-valued impulse response coefficients
- These filters are employed in a number of practical applications, primarily because of their simplicity, which makes them amenable to inexpensive hardware implementations

Simple FIR Digital Filters

Lowpass FIR Digital Filters

- The simplest lowpass FIR digital filter is the 2-point moving-average filter given by

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1}) = \frac{z+1}{2z}$$

- The above transfer function has a zero at $z = -1$ and a pole at $z = 0$
- Note that here the pole vector has a unity magnitude for all values of ω

Simple FIR Digital Filters

- On the other hand, as ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2, the diameter of the unit circle, to 0
- Hence, the magnitude response $|H_0(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$

Simple FIR Digital Filters

- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$, i.e.,

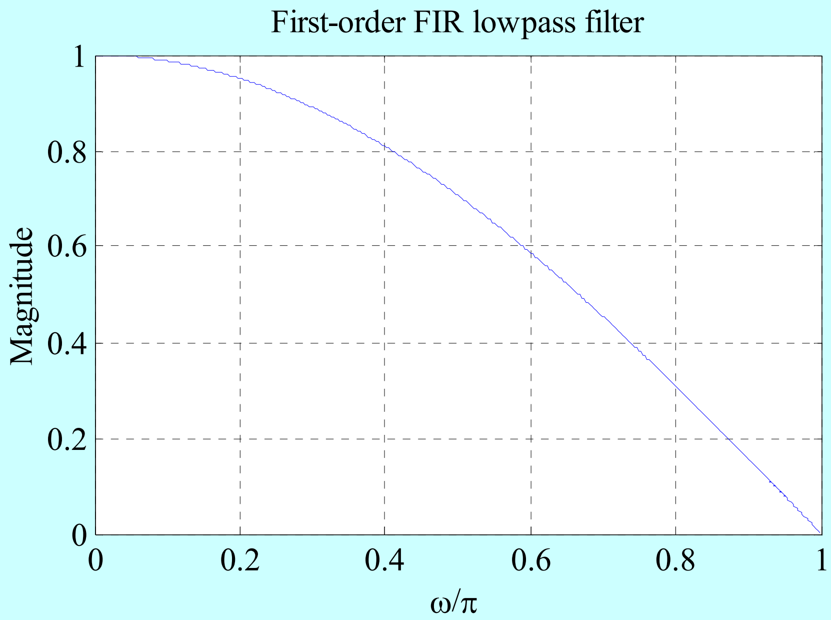
$$|H_0(e^{j0})| = 1, \quad |H_0(e^{j\pi})| = 0$$

- The frequency response of the above filter is given by

$$H_0(e^{j\omega}) = e^{-j\omega/2}\cos(\omega/2)$$

Simple FIR Digital Filters

- The magnitude response $|H_0(e^{j\omega})| = \cos(\omega/2)$ is a monotonically decreasing function of ω



Simple FIR Digital Filters

- The frequency $\omega = \omega_c$ at which

$$\left|H_0(e^{j\omega_c})\right| = \frac{1}{\sqrt{2}}\left|H_0(e^{j0})\right|$$

is of practical interest since here the gain in dB is

$$\mathrm{G}(\omega_c) = 20\log_{10}\left|H(e^{j\omega_c})\right|$$

$$= 20\log_{10}\left|H(e^{j0})\right| - 20\log_{10}\sqrt{2} \cong -3 \text{ dB}$$

since the DC gain is

$$20\log_{10}\left|H(e^{j0})\right| = 0$$

Simple FIR Digital Filters

- Thus, the gain $G(\omega)$ at $\omega = \omega_c$ is approximately 3 dB less than the gain at $\omega=0$
- As a result, ω_c is called the **3-dB cutoff frequency**
- To determine the value of ω_c we set

$$|H_0(e^{j\omega_c})|^2 = \cos^2(\omega_c / 2) = \frac{1}{2}$$

which yields $\omega_c = \pi / 2$

Simple FIR Digital Filters

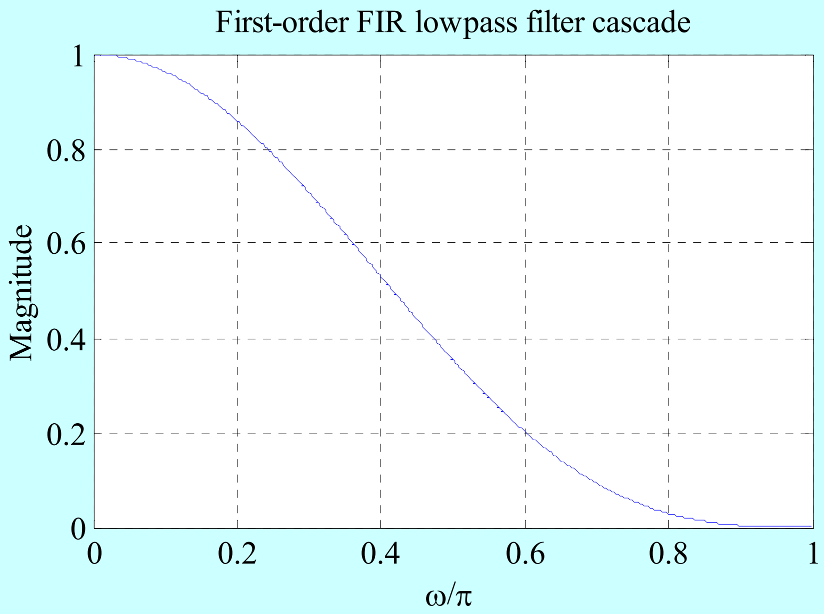
- The 3-dB cutoff frequency ω_c can be considered as the passband edge frequency
- As a result, for the filter $H_0(z)$ the passband width is approximately $\pi/2$
- The stopband is from $\pi/2$ to π
- Note: $H_0(z)$ has a zero at $z = -1$ or $\omega = \pi$, which is in the stopband of the filter

Simple FIR Digital Filters

- A cascade of the simple FIR filter

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1})$$

results in an improved lowpass frequency response as illustrated below for a cascade of 3 sections



Simple FIR Digital Filters

- The 3-dB cutoff frequency of a cascade of M sections is given by

$$\omega_c = 2\cos^{-1}(2^{-1/2M})$$

- For $M = 3$, the above yields $\omega_c = 0.302\pi$
- Thus, the cascade of first-order sections yields a sharper magnitude response but at the expense of a decrease in the width of the passband

Simple FIR Digital Filters

- A better approximation to the ideal lowpass filter is given by a higher-order Moving Average (MA) filter
- Signals with rapid fluctuations in sample values are generally associated with high-frequency components
- These high-frequency components are essentially removed by an MA filter resulting in a smoother output waveform

Simple FIR Digital Filters

Highpass FIR Digital Filters

- The simplest highpass FIR filter is obtained from the simplest lowpass FIR filter by replacing z with $-z$
- This results in

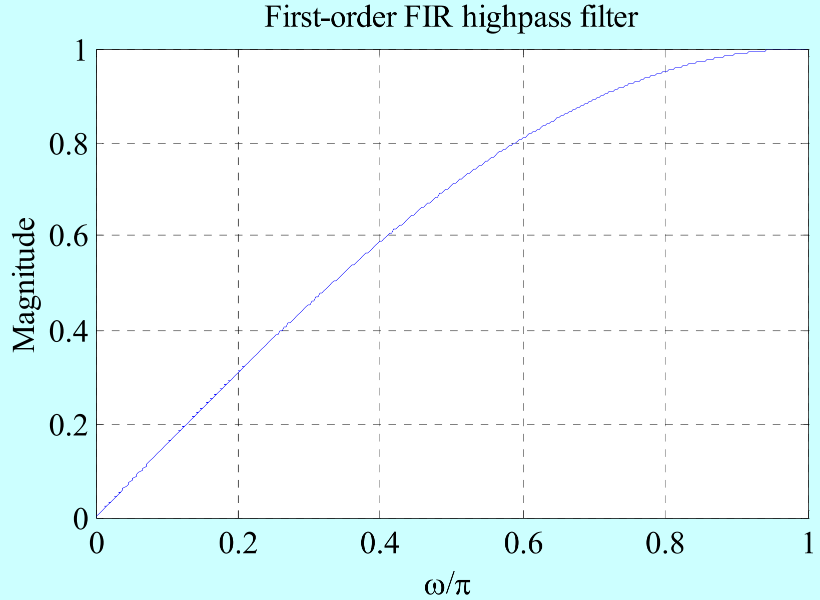
$$H_1(z) = \tfrac{1}{2}(1 - z^{-1})$$

Simple FIR Digital Filters

- Corresponding frequency response is given by

$$H_1(e^{j\omega}) = j\,e^{-j\omega/2}\sin(\omega/2)$$

whose magnitude response is plotted below



Simple FIR Digital Filters

- The monotonically increasing behavior of the magnitude function can again be demonstrated by examining the pole-zero pattern of the transfer function $H_1(z)$
- The highpass transfer function $H_1(z)$ has a zero at $z = 1$ or $\omega = 0$ which is in the stopband of the filter

Simple FIR Digital Filters

- Improved highpass magnitude response can again be obtained by cascading several sections of the first-order highpass filter
- Alternately, a higher-order highpass filter of the form

$$H_1(z) = \frac{1}{M} \sum_{n=0}^{M-1} (-1)^n z^{-n}$$

is obtained by replacing z with $-z$ in the transfer function of an MA filter

Simple IIR Digital Filters

Lowpass IIR Digital Filters

- A first-order causal lowpass IIR digital filter has a transfer function given by

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

where $|\alpha| < 1$ for stability

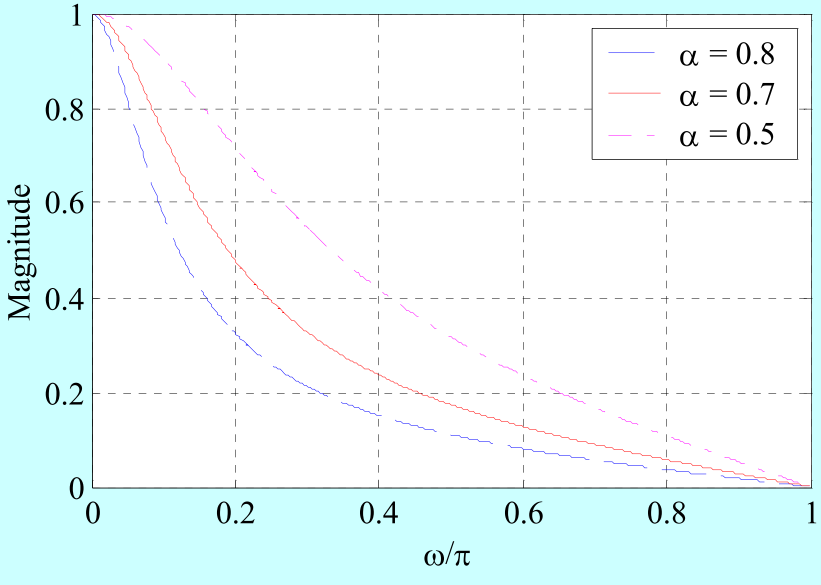
- The above transfer function has a zero at $z = -1$ i.e., at $\omega = \pi$ which is in the stopband

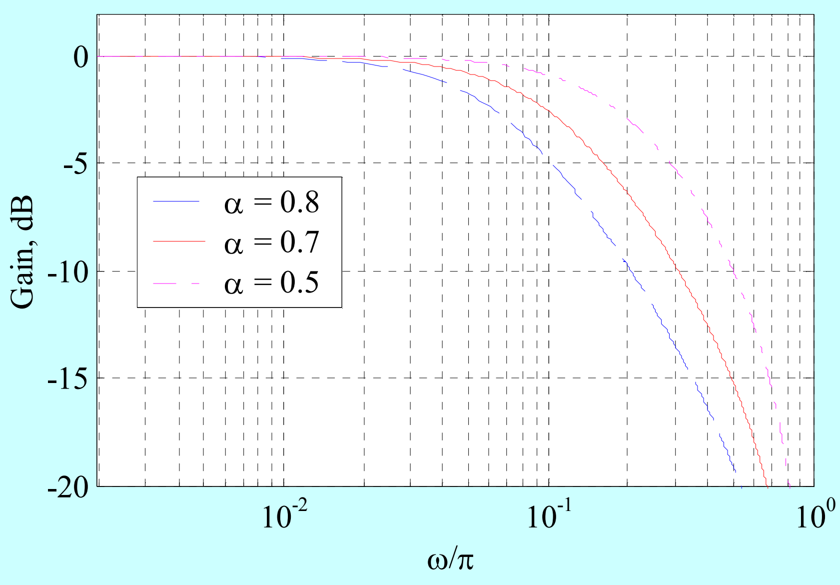
Simple IIR Digital Filters

- $H_{LP}(z)$ has a real pole at $z = \alpha$
- As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2 to 0, whereas, for a positive value of α, the magnitude of the pole vector increases from a value of $1-\alpha$ to $1+\alpha$
- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$

Simple IIR Digital Filters

- i.e., $|H_{LP}(e^{j0})| = 1, \qquad |H_{LP}(e^{j\pi})| = 0$
- Therefore, $|H_{LP}(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$ as indicated below





Simple IIR Digital Filters

- The squared magnitude function is given by

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2 - 2\alpha\cos\omega)}$$

- The derivative of $|H_{LP}(e^{j\omega})|^2$ with respect to ω is given by

$$\frac{d\,|H_{LP}(e^{j\omega})|^2}{d\omega} = \frac{-(1-\alpha)^2(1+2\alpha+\alpha^2)\sin\omega}{2(1-2\alpha\cos\omega+\alpha^2)^2}$$

Simple IIR Digital Filters

$d\left|H_{LP}(e^{j\omega})\right|^2 / d\omega \leq 0$ in the range $0 \leq \omega \leq \pi$ verifying again the monotonically decreasing behavior of the magnitude function

- To determine the 3-dB cutoff frequency we set

$$\left|H_{LP}(e^{j\omega_c})\right|^2 = \frac{1}{2}$$

in the expression for the squared magnitude function resulting in

Simple IIR Digital Filters

$$\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)}=\frac{1}{2}$$

or

$$(1-\alpha)^2(1+\cos\omega_c)=1+\alpha^2-2\alpha\cos\omega_c$$

which when solved yields

$$\cos\omega_c=\frac{2\alpha}{1+\alpha^2}$$

- The above quadratic equation can be solved for α yielding two solutions

Simple IIR Digital Filters

- The solution resulting in a stable transfer function $H_{LP}(z)$ is given by

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

- It follows from

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

that $H_{LP}(z)$ is a BR function for $|\alpha| < 1$

Simple IIR Digital Filters

Highpass IIR Digital Filters

- A first-order causal highpass IIR digital filter has a transfer function given by

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

where $|\alpha| < 1$ for stability

- The above transfer function has a zero at $z = 1$ i.e., at $\omega = 0$ which is in the stopband
- It is a BR function for $|\alpha| < 1$

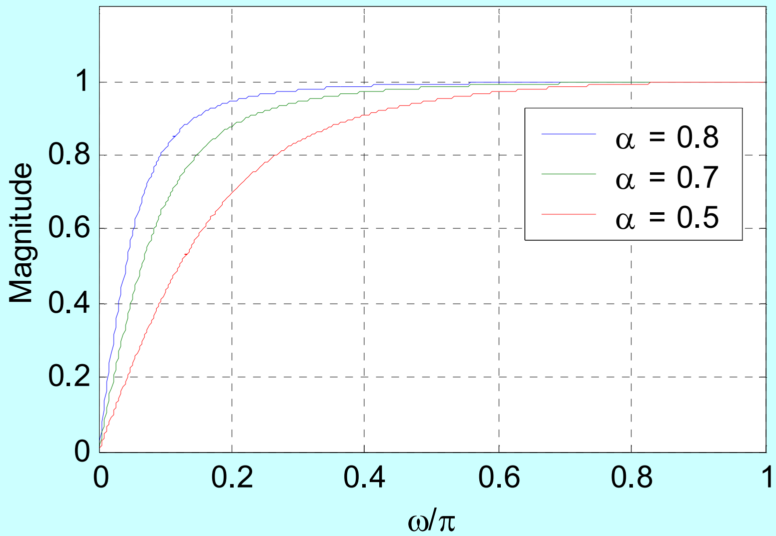
Simple IIR Digital Filters

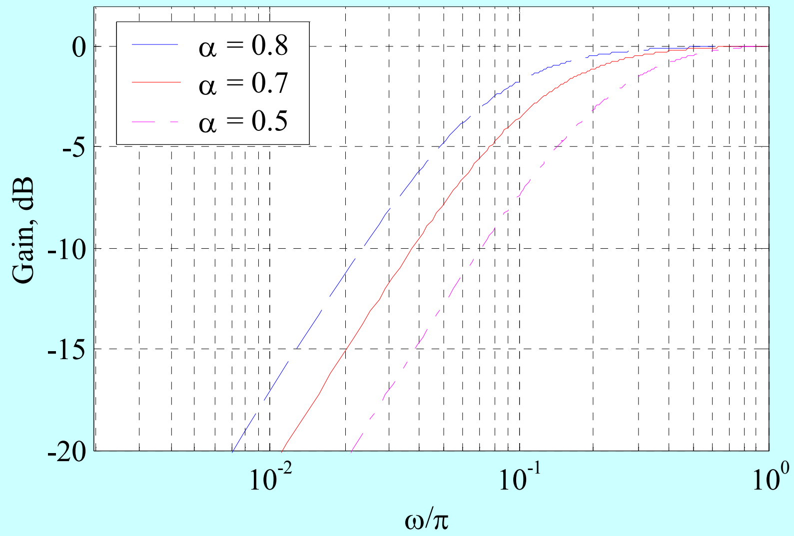
- Its 3-dB cutoff frequency ω_c is given by

$$\alpha = (1 - \sin\omega_c)/\cos\omega_c$$

which is the same as that of $H_{LP}(z)$

- Magnitude and gain responses of $H_{HP}(z)$ are shown below





Example 1-First Order HP Filter

- Design a first-order highpass filter with a 3-dB cutoff frequency of 0.8π

- Now, $\sin(\omega_c) = \sin(0.8\pi) = 0.587785$
  and $\cos(0.8\pi) = -0.80902$

- Therefore

$$\alpha = (1 - \sin\omega_c)/\cos\omega_c = -0.5095245$$

Example 1-First Order HP Filter

- Therefore,

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

$$= 0.245238\left(\frac{1-z^{-1}}{1+0.5095245\,z^{-1}}\right)$$

Simple IIR Digital Filters

Bandpass IIR Digital Filters

- A 2nd-order bandpass digital transfer function is given by

$$H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- Its squared magnitude function is

$$\left|H_{BP}(e^{j\omega})\right|^2$$

$$= \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1+\beta^2(1+\alpha)^2+\alpha^2-2\beta(1+\alpha)^2\cos\omega+2\alpha\cos 2\omega]}$$

Simple IIR Digital Filters

- $|H_{BP}(e^{j\omega})|^2$ goes to zero at $\omega = 0$ and $\omega = \pi$
- It assumes a maximum value of 1 at $\omega = \omega_o$, called the **center frequency** of the bandpass filter, where

$$\omega_o = \cos^{-1}(\beta)$$

- The frequencies ω_{c1} and ω_{c2} where $|H_{BP}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**

Simple IIR Digital Filters

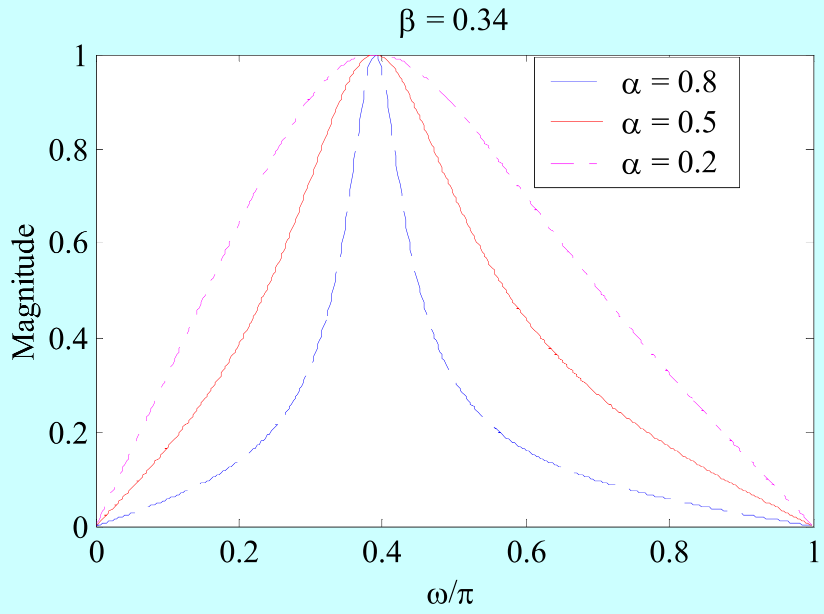
- The difference between the two cutoff frequencies, assuming $\omega_{c2} > \omega_{c1}$ is called the **3-dB bandwidth** and is given by

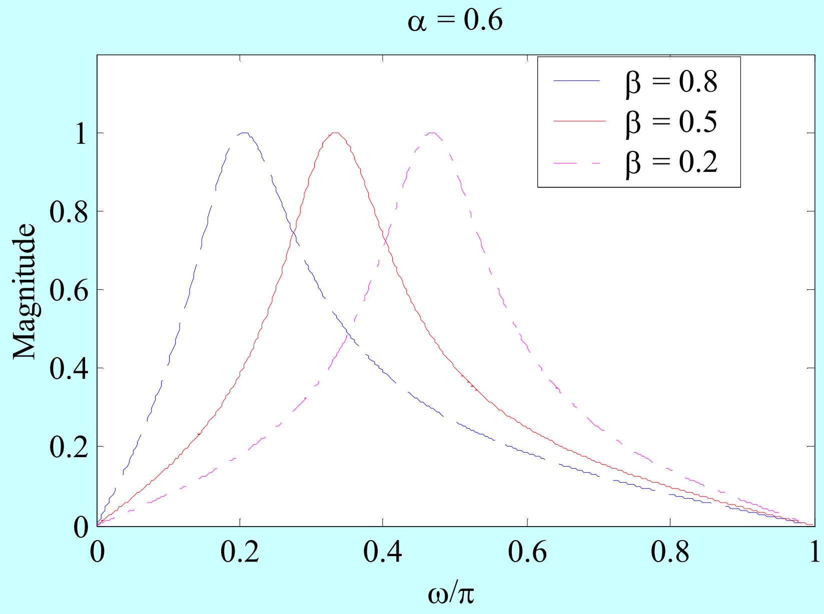
$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

- The transfer function $H_{BP}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$

Simple IIR Digital Filters

- Plots of $|H_{BP}(e^{j\omega})|$ are shown below





Example 2-Second Order BP Filter

- Design a 2nd order bandpass digital filter with center frequency at 0.4π and a 3-dB bandwidth of 0.1π
- Here $\beta = \cos(\omega_o) = \cos(0.4\pi) = 0.309017$

  and

  $$\frac{2\alpha}{1+\alpha^2} = \cos(B_w) = \cos(0.1\pi) = 0.9510565$$

- The solution of the above equation yields: $\alpha = 1.376382$ and $\alpha = 0.72654253$

Example 2-Second Order BP Filter

- The corresponding transfer functions are

$$H'_{BP}(z) = -0.18819 \frac{1 - z^{-2}}{1 - 0.7343424 z^{-1} + 1.37638 z^{-2}}$$

and

$$H''_{BP}(z) = 0.13673 \frac{1 - z^{-2}}{1 - 0.533531 z^{-1} + 0.72654253 z^{-2}}$$

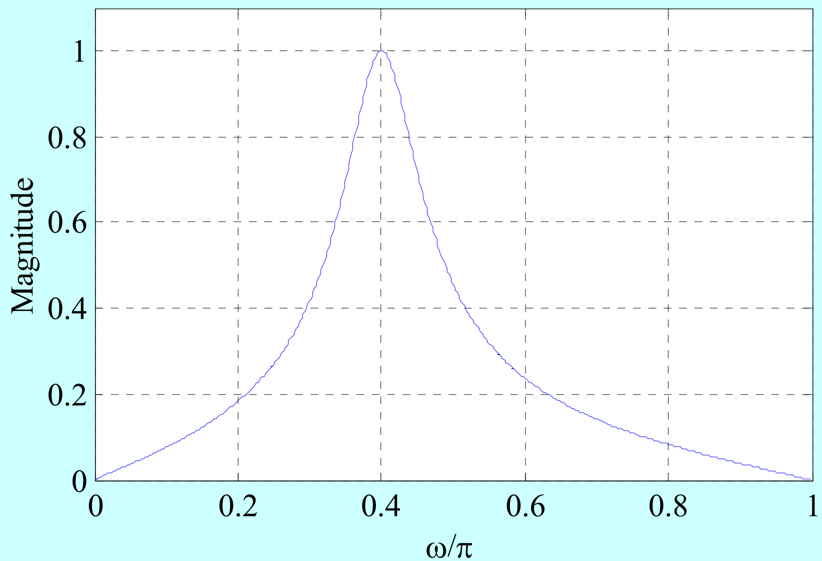
- The poles of $H'_{BP}(z)$ are at $z = 0.3671712 \pm j1.11425636$ and have a magnitude > 1

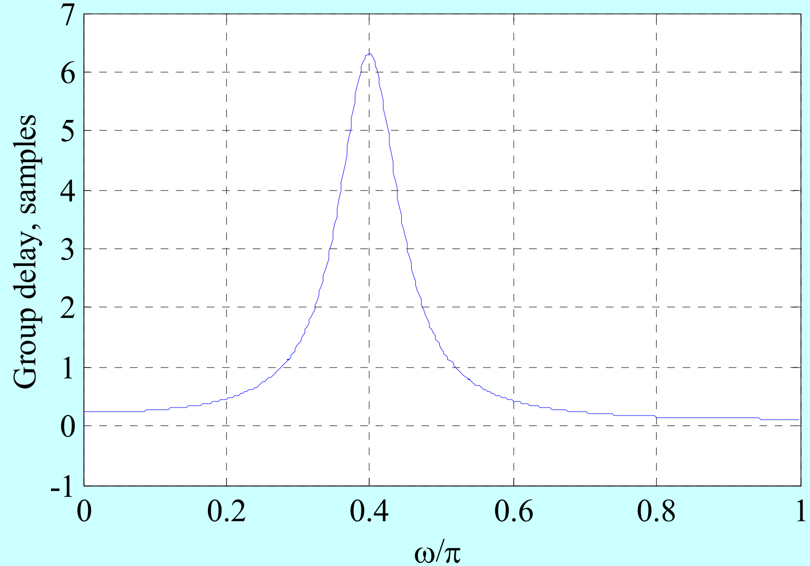
Example 2-Second Order BP Filter

- Thus, the poles of $H'_{BP}(z)$ are outside the unit circle making the transfer function unstable
- On the other hand, the poles of $H''_{BP}(z)$ are at $z = 0.2667655 \pm j0.8095546$ and have a magnitude of 0.8523746
- Hence, $H''_{BP}(z)$ is BIBO stable

Example 2-Second Order BP Filter

- Figures below show the plots of the magnitude function and the group delay of $H''_{BP}(z)$





Simple IIR Digital Filters

Bandstop IIR Digital Filters

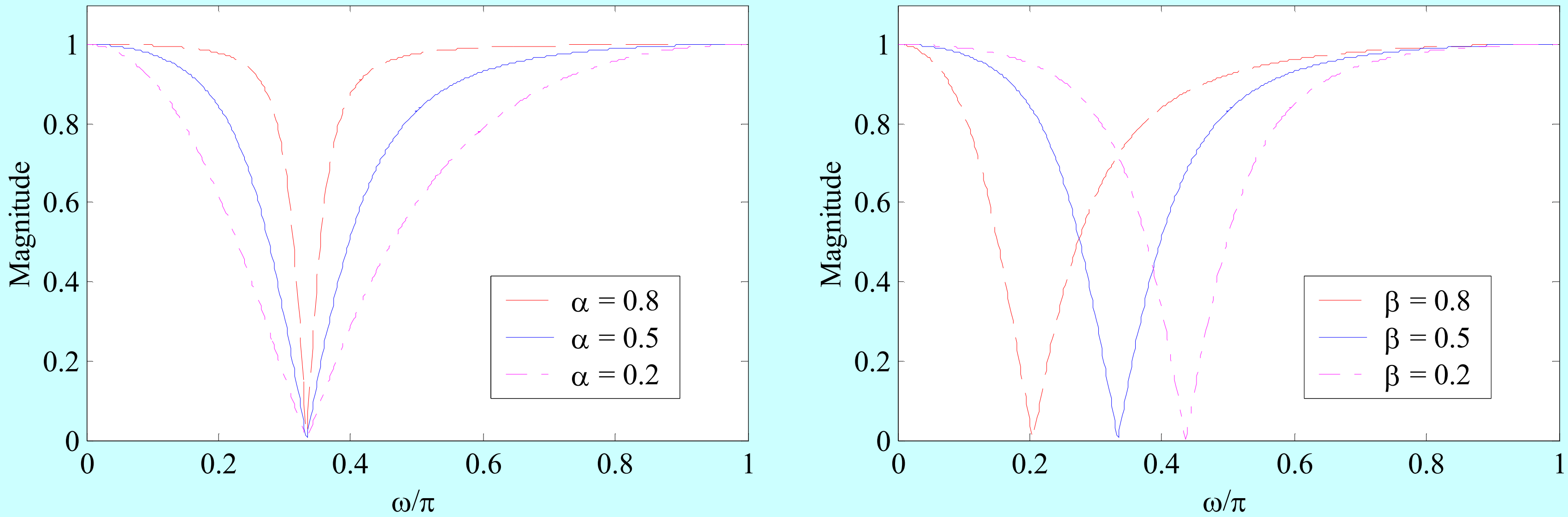
- A 2nd-order bandstop digital filter has a transfer function given by

$$H_{BS}(z) = \frac{1+\alpha}{2}\left(\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- The transfer function $H_{BS}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$

Simple IIR Digital Filters

- Its magnitude response is plotted below



Simple IIR Digital Filters

- Here, the magnitude function takes the maximum value of 1 at $\omega = 0$ and $\omega = \pi$
- It goes to 0 at $\omega = \omega_o$, where ω_o , called the **notch frequency**, is given by

$$\omega_o = \cos^{-1}(\beta)$$

- The digital transfer function $H_{BS}(z)$ is more commonly called a **notch filter**

Simple IIR Digital Filters

- The frequencies ω_{c1} and ω_{c2} where $|H_{BS}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**
- The difference between the two cutoff frequencies, assuming $\omega_{c2} > \omega_{c1}$ is called the **3-dB notch bandwidth** and is given by

$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

Simple IIR Digital Filters

Higher-Order IIR Digital Filters

- By cascading the simple digital filters discussed so far, we can implement digital filters with sharper magnitude responses
- Consider a cascade of K first-order lowpass sections characterized by the transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

Simple IIR Digital Filters

- The overall structure has a transfer function given by

$$G_{LP}(z) = \left(\frac{1-\alpha}{2} \frac{1+z^{-1}}{1-\alpha z^{-1}} \right)^K$$

- The corresponding squared-magnitude function is given by

$$|G_{LP}(e^{j\omega})|^2 = \left[\frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)} \right]^K$$

Simple IIR Digital Filters

- To determine the relation between its 3-dB cutoff frequency ω_c and the parameter α, we set

$$\left[\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)}\right]^K=\frac{1}{2}$$

which when solved for α, yields for a stable $G_{LP}(z)$:

$$\alpha=\frac{1+(1-C)\cos\omega_c-\sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c}$$

Simple IIR Digital Filters

where

$$C = 2^{(K-1)/K}$$

- It should be noted that the expression given above reduces to

$$\alpha = \frac{1 - \sin\omega_c}{\cos\omega_c}$$

for $K = 1$

Example 3-Design of an LP Filter

- Design a lowpass filter with a 3-dB cutoff frequency at $\omega_c = 0.4\pi$ using a single first-order section and a cascade of 4 first-order sections, and compare their gain responses
- For the single first-order lowpass filter we have

$$\alpha = \frac{1+\sin\omega_c}{\cos\omega_c} = \frac{1+\sin(0.4\pi)}{\cos(0.4\pi)} = 0.1584$$

Example 3-Design of an LP Filter

- For the cascade of 4 first-order sections, we substitute $K = 4$ and get

$$C = 2^{(K-1)/K} = 2^{(4-1)/4} = 1.6818$$

- Next we compute

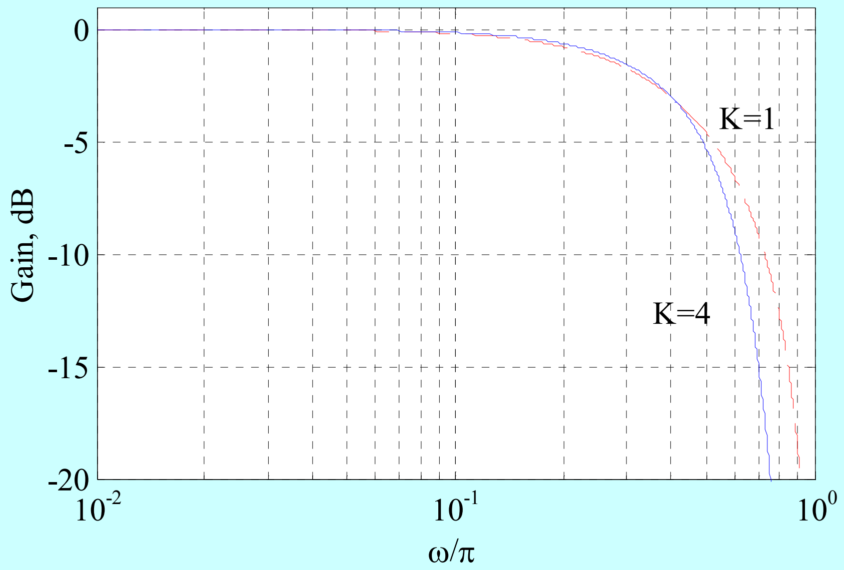
$$\alpha = \frac{1+(1-C)\cos\omega_c - \sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c}$$

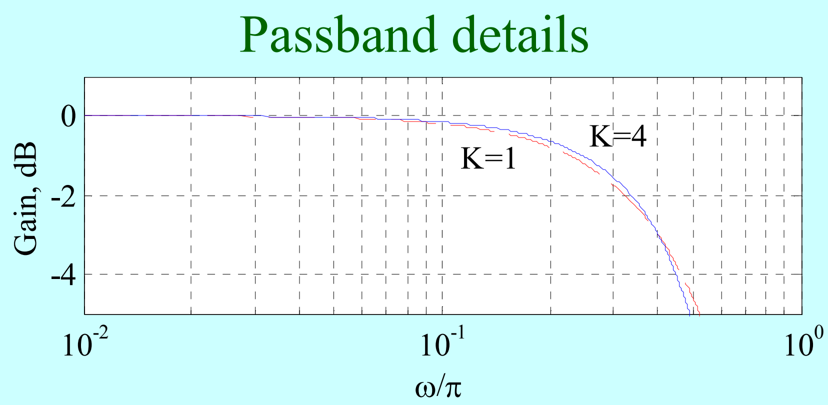
$$= \frac{1+(1-1.6818)\cos(0.4\pi)-\sin(0.4\pi)\sqrt{2(1.6818)-(1.6818)^2}}{1-1.6818+\cos(0.4\pi)}$$

$$= -0.251$$

Example 3-Design of an LP Filter

- The gain responses of the two filters are shown below
- As can be seen, cascading has resulted in a sharper roll-off in the gain response





Comb Filters

- The simple filters discussed so far are characterized either by a single passband and/or a single stopband

- There are applications where filters with multiple passbands and stopbands are required

- The **comb filter** is an example of such filters

Comb Filters

- In its most general form, a comb filter has a frequency response that is a periodic function of ω with a period $2\pi/L$, where L is a positive integer
- If $H(z)$ is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with L delays resulting in a structure with a transfer function given by $G(z) = H(z^L)$

Comb Filters

- If $|H(e^{j\omega})|$ exhibits a peak at ω_p, then $|G(e^{j\omega})|$ will exhibit L peaks at $\omega_p k/L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega < 2\pi$
- Likewise, if $|H(e^{j\omega})|$ has a notch at ω_o, then $|G(e^{j\omega})|$ will have L notches at $\omega_o k/L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega < 2\pi$
- A comb filter can be generated from either an FIR or an IIR prototype filter

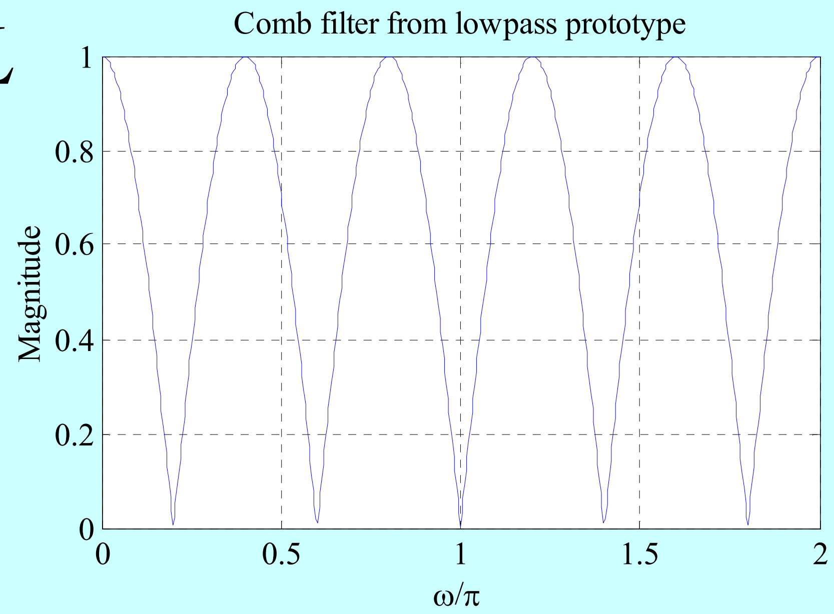
Comb Filters

- For example, the comb filter generated from the prototype lowpass FIR filter $H_0(z) = \frac{1}{2}(1 + z^{-1})$ has a transfer function

$$G_0(z) = H_0(z^L) = \frac{1}{2}(1 + z^{-L})$$

- $|G_0(e^{j\omega})|$ has L notches at $\omega = (2k+1)\pi/L$ and L peaks at $\omega = 2\pi\, k/L$,

$0 \le k \le L - 1$, in the frequency range $0 \le \omega < 2\pi$

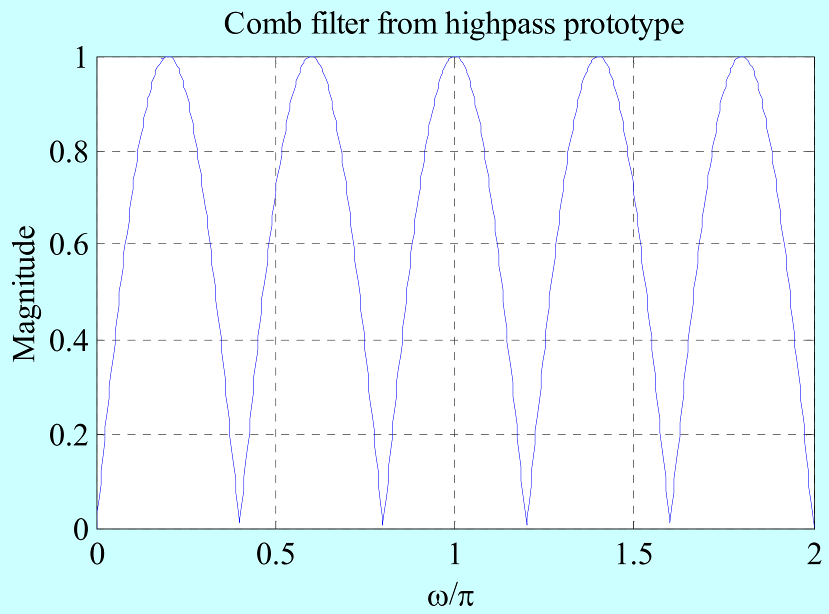


Comb Filters

- Furthermore, the comb filter generated from the prototype highpass FIR filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$ has a transfer function

$$G_1(z) = H_1(z^L) = \frac{1}{2}(1 - z^{-L})$$

- $|G_1(e^{j\omega})|$ has L peaks at $\omega = (2k+1)\pi/L$ and L notches at $\omega = 2\pi\, k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$



Comb Filters

- Depending on applications, comb filters with other types of periodic magnitude responses can be easily generated by appropriately choosing the prototype filter
- For example, the M-point moving average filter

$$H(z) = \frac{1 - z^{-M}}{M(1 - z^{-1})}$$

has been used as a prototype

Comb Filters

- This filter has a peak magnitude at $\omega = 0$, and $M-1$ notches at $\omega = 2\pi\ell/M$, $1 \le \ell \le M-1$
- The corresponding comb filter has a transfer function

$$G(z) = \frac{1 - z^{-LM}}{M(1 - z^{-L})}$$

whose magnitude has L peaks at $\omega = 2\pi k/L$, $0 \le k \le L-1$ and $L(M-1)$ notches at $\omega = 2\pi k/LM$, $1 \le k \le L(M-1)$

Allpass Transfer Functions

Definition

- An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

$$|A(e^{j\omega})|^2 = 1, \qquad \text{for all } \omega$$

is called an **allpass transfer function**

- An M-th order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \ldots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \ldots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

Allpass Transfer Functions

- If we denote the denominator polynomials of $A_M(z)$ as $D_M(z)$:

  $D_M(z) = 1 + d_1 z^{-1} + \ldots + d_{M-1} z^{-M+1} + d_M z^{-M}$

  then it follows that $A_M(z)$ can be written as:

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = \frac{1}{r} e^{-j\phi}$

Allpass Transfer Functions

- The numerator of a real-coefficient allpass transfer function is said to be the **mirror-image polynomial** of the denominator, and vice versa
- We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-M polynomial $D_M(z)$, i.e.,

$$\tilde{D}_M(z) = z^{-M} D_M(z)$$

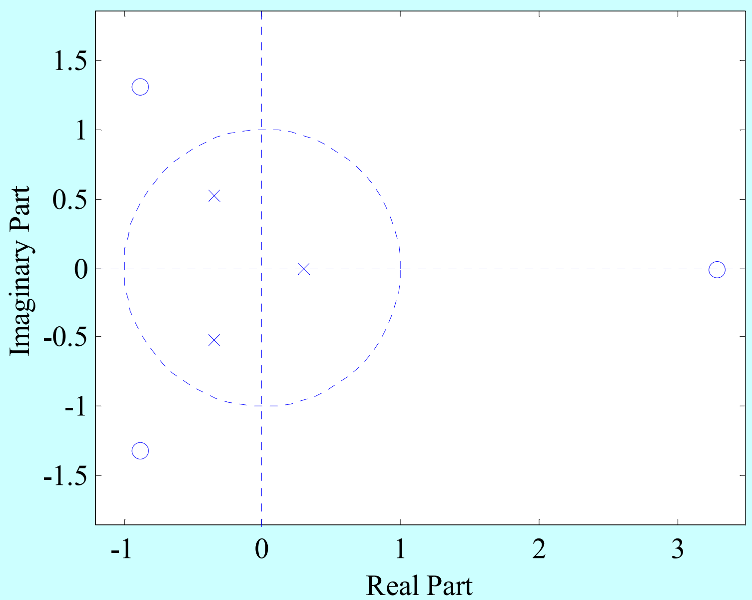
Allpass Transfer Functions

- The expression

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

implies that the poles and zeros of a real-coefficient allpass function exhibit **mirror-image symmetry** in the *z*-plane

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$



Allpass Transfer Functions

- To show that $|A_M(e^{j\omega})| = 1$ we observe that

$$A_M(z^{-1}) = \pm \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Therefore

$$A_M(z) A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Hence, $|A_M(e^{j\omega})|^2 = A_M(z) A_M(z^{-1})\big|_{z=e^{j\omega}} = 1$

Allpass Transfer Functions

- Now, the poles of a causal stable transfer function must lie inside the unit circle in the z-plane
- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a **mirror-image symmetry** with its poles situated inside the unit circle
- A causal stable real-coefficient allpass transfer function is a **lossless bounded real (LBR)** function or, equivalently, a causal stable allpass filter is a lossless structure

Allpass Transfer Functions

- The magnitude function of a stable allpass function $A(z)$ satisfies:

$$|A(z)| \begin{cases} <1, & \text{for } |z|>1 \\ =1, & \text{for } |z|=1 \\ >1, & \text{for } |z|<1 \end{cases}$$

- Let $\tau(\omega)$ denote the group delay function of an allpass filter $A(z)$, i.e.,

$$\tau(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

Allpass Transfer Functions

- The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of ω so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$
- The group delay of an M-th order stable real-coefficient allpass transfer function satisfies:

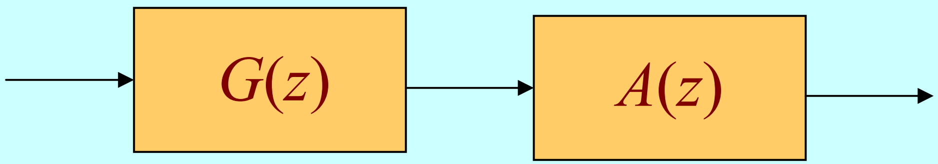
$$\int_0^{\pi} \tau(\omega) d\omega = M\pi$$

Allpass Transfer Function

A Simple Application

- A simple but often used application of an allpass filter is as a **delay equalizer**
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest

Allpass Transfer Function



- Since $|A(e^{j\omega})| = 1$, we have

$$|G(e^{j\omega})A(e^{j\omega})| = |G(e^{j\omega})|$$

- Overall group delay is the given by the sum of the group delays of $G(z)$ and $A(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

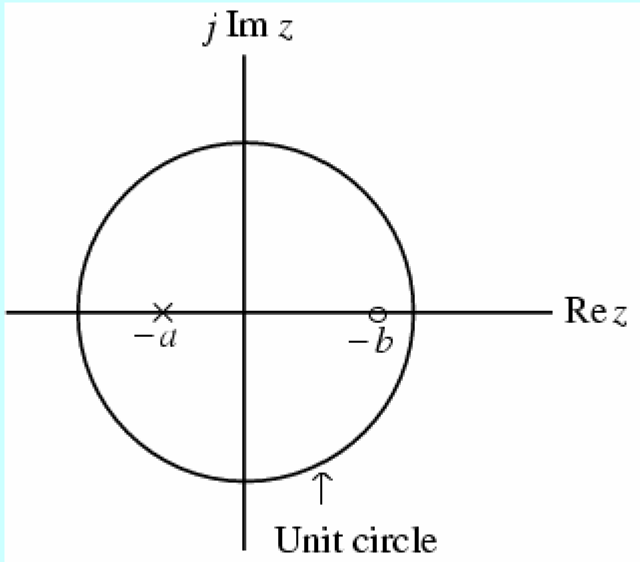
- Consider the two 1st-order transfer functions:

$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1$$

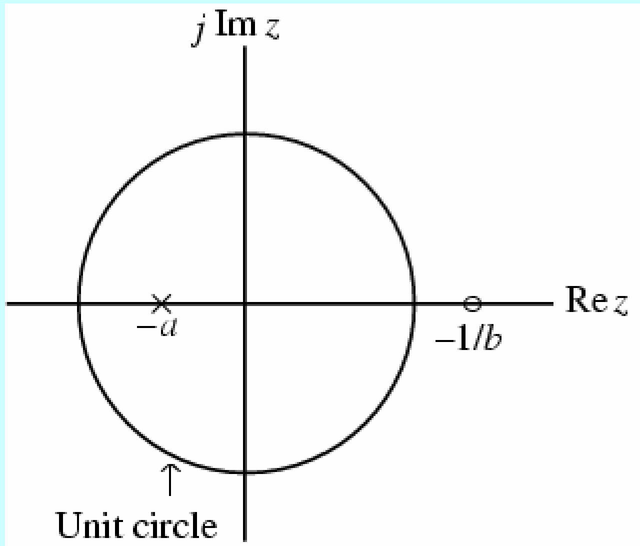
- Both transfer functions have a pole inside the unit circle at the same location $z = -a$ and are stable
- But the zero of $H_1(z)$ is inside the unit circle at $z = -b$, whereas, the zero of $H_2(z)$ is at $z = -\frac{1}{b}$ situated in a mirror-image symmetry

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the pole-zero plots of the two transfer functions



$H_1(z)$



$H_2(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

- However, both transfer functions have an identical magnitude as

$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1})$$

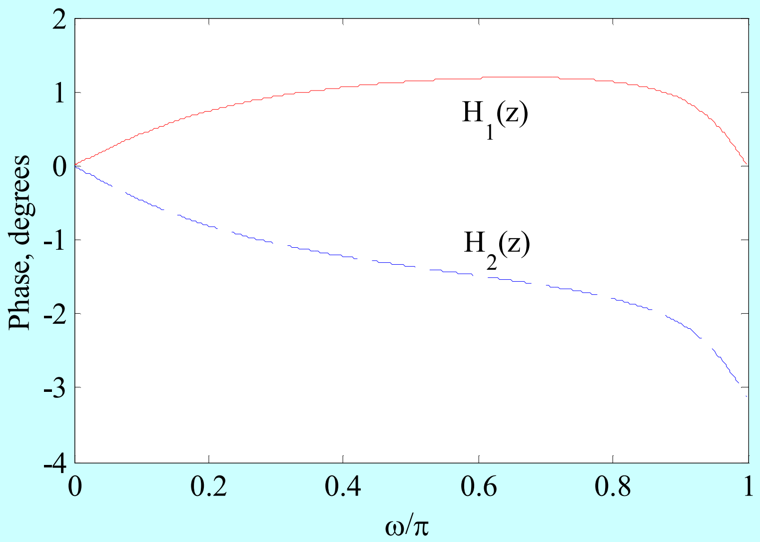
- The corresponding phase functions are

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the unwrapped phase responses of the two transfer functions for a=0.8 and b=-0.5



Minimum-Phase and Maximum-Phase Transfer Functions

- From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$
- Generalizing the above result, we can show that a causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal transfer function with identical magnitude but having all zeros inside the unit circle

Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros inside the unit circle is called a **minimum-phase transfer function**
- A causal stable transfer function with all zeros outside the unit circle is called a **maximum-phase transfer function**
- Any nonminimum-phase transfer function can be expressed as the product of a minimum-phase transfer function and a stable allpass transfer function

Complementary Transfer Functions

- A set of digital transfer functions with complementary characteristics often finds useful applications in practice
- Four useful complementary relations are described next along with some applications

Complementary Transfer Functions

Delay-Complementary Transfer Functions

- A set of L transfer functions, $\{H_i(z)\}$, $0 \leq i \leq L-1$, is defined to be **delay-complementary** of each other if the sum of their transfer functions is equal to some integer multiple of unit delays, i.e.,

$$\sum_{i=0}^{L-1} H_i(z) = \beta z^{-n_o}, \qquad \beta \neq 0$$

where n_o is a nonnegative integer

Complementary Transfer Functions

- A delay-complementary pair $\{H_0(z), H_1(z)\}$ can be readily designed if one of the pairs is a known Type 1 FIR transfer function of odd length
- Let $H_0(z)$ be a Type 1 FIR transfer function of length $M = 2K+1$
- Then its delay-complementary transfer function is given by

$$H_1(z) = z^{-K} - H_0(z)$$

Complementary Transfer Functions

- Let the magnitude response of $H_0(z)$ be equal to $1 \pm \delta_p$ in the passband and less than or equal to δ_s in the stopband where δ_p and δ_s are very small numbers
- Now the frequency response of $H_0(z)$ can be expressed as

$$H_0(e^{j\omega}) = e^{-jK\omega}\tilde{H}_0(\omega)$$

where $\tilde{H}_0(\omega)$ is the **amplitude response**

Complementary Transfer Functions

- Its delay-complementary transfer function $H_1(z)$ has a frequency response given by

$$H_1(e^{j\omega}) = e^{-jK\omega}\tilde{H}_1(\omega) = e^{-jK\omega}[1 - \tilde{H}_0(\omega)]$$

- Now, in the passband, $1-\delta_p \le \tilde{H}_0(\omega) \le 1+\delta_p$, and in the stopband, $-\delta_s \le \tilde{H}_0(\omega) \le \delta_s$
- It follows from the above equation that in the passband, $-\delta_p \le \tilde{H}_1(\omega) \le \delta_p$ and in the stopband, $1-\delta_s \le \tilde{H}_1(\omega) \le 1+\delta_s$

Complementary Transfer Functions

- As a result, $H_1(z)$ has a complementary magnitude response characteristic, with a stopband exactly identical to the passband of $H_0(z)$, and a passband that is exactly identical to the stopband of $H_0(z)$
- Thus, if $H_0(z)$ is a lowpass filter, $H_1(z)$ will be a highpass filter, and vice versa

Complementary Transfer Functions

- At frequency ω_o at which

$$\tilde{H}_0(\omega_o) = \tilde{H}_1(\omega_o) = 0.5$$

  the gain responses of both filters are 6 dB below their maximum values

- The frequency ω_o is thus called the **6-dB crossover frequency**

Example 4

- Consider the Type 1 bandstop transfer function

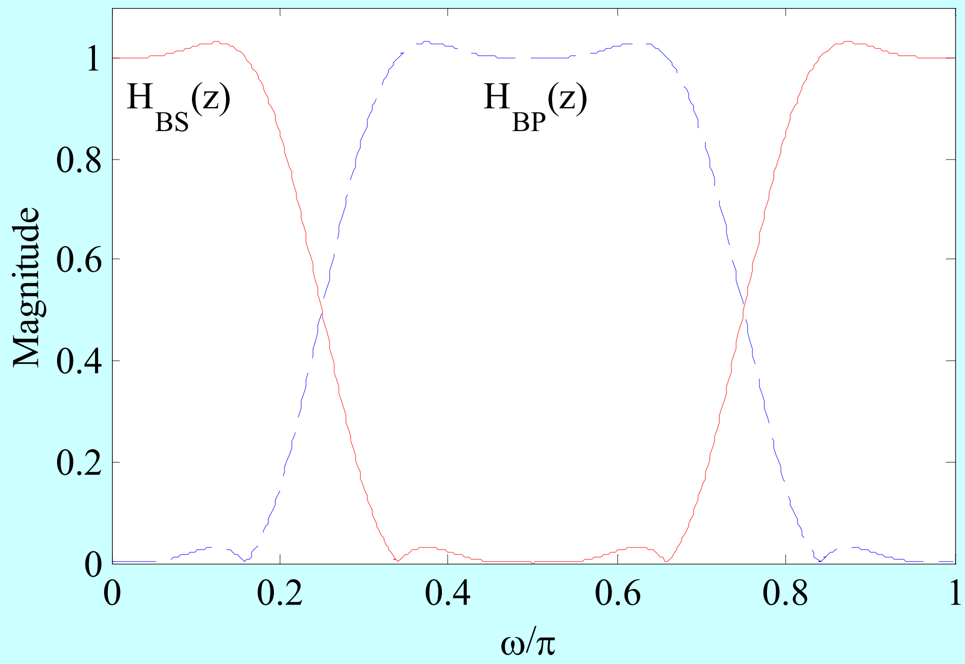
$$H_{BS}(z) = \frac{1}{64}(1+z^{-2})^4(1-4z^{-2}+5z^{-4}+5z^{-8}-4z^{-10}+z^{-12})$$

- Its delay-complementary Type 1 bandpass transfer function is given by

$$H_{BP}(z) = z^{-10} - H_{BS}(z)$$
$$= -\frac{1}{64}(1-z^{-2})^4(1+4z^{-2}+5z^{-4}+5z^{-8}+4z^{-10}+z^{-12})$$

Example 4

- Plots of the magnitude responses of $H_{BS}(z)$ and $H_{BP}(z)$ are shown below



Complementary Transfer Functions

Allpass Complementary Filters

- A set of M digital transfer functions, $\{H_i(z)\}$, $0 \leq i \leq M-1$, is defined to be **allpass-complementary** of each other, if the sum of their transfer functions is equal to an allpass function, i.e.,

$$\sum_{i=0}^{M-1} H_i(z) = A(z)$$

Complementary Transfer Functions

Power-Complementary Transfer Functions

- A set of M digital transfer functions, $\{H_i(z)\}$, $0 \leq i \leq M-1$, is defined to be **power-complementary** of each other, if the sum of their square-magnitude responses is equal to a constant K for all values of ω, i.e.,

$$\sum_{i=0}^{M-1} \left| H_i(e^{j\omega}) \right|^2 = K, \qquad \text{for all } \omega$$

Complementary Transfer Functions

- By analytic continuation, the above property is equal to

$$\sum_{i=0}^{M-1} H_i(z) H_i(z^{-1}) = K, \qquad \text{for all } \omega$$

for real coefficient $H_i(z)$

- Usually, by scaling the transfer functions, the power-complementary property is defined for $K = 1$

Complementary Transfer Functions

- For a pair of power-complementary transfer functions, $H_0(z)$ and $H_1(z)$, the frequency ω_o where $|H_0(e^{j\omega_o})|^2 = |H_1(e^{j\omega_o})|^2 = 0.5$, is called the **cross-over frequency**
- At this frequency the gain responses of both filters are 3-dB below their maximum values
- As a result, ω_o is called the **3-dB cross-over frequency**

Complementary Transfer Functions

- Consider the two transfer functions $H_0(z)$ and $H_1(z)$ given by

$$H_0(z) = \tfrac{1}{2}[A_0(z) + A_1(z)]$$
$$H_1(z) = \tfrac{1}{2}[A_0(z) - A_1(z)]$$

where $A_0(z)$ and $A_1(z)$ are stable allpass transfer functions

- Note that $H_0(z) + H_1(z) = A_0(z)$
- Hence, $H_0(z)$ and $H_1(z)$ are allpass complementary

Complementary Transfer Functions

- It can be shown that $H_0(z)$ and $H_1(z)$ are also power-complementary
- Moreover, $H_0(z)$ and $H_1(z)$ are bounded-real transfer functions

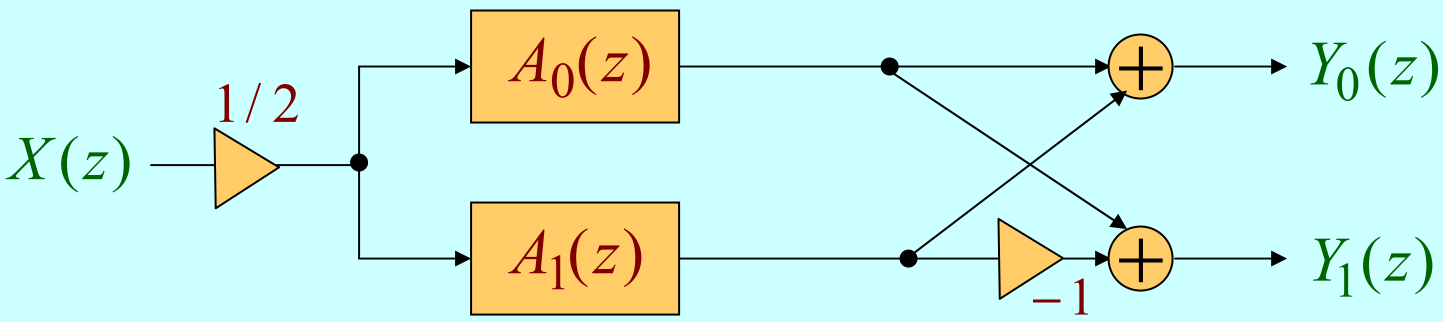
Complementary Transfer Functions

Doubly-Complementary Transfer Functions

- A set of M transfer functions satisfying both the allpass complementary and the power-complementary properties is known as a **doubly-complementary** set

Complementary Transfer Functions

- A pair of doubly-complementary IIR transfer functions, $H_0(z)$ and $H_1(z)$, with a sum of allpass decomposition can be simply realized as indicated below



$$H_0(z) = \frac{Y_0(z)}{X(z)} \qquad H_1(z) = \frac{Y_1(z)}{X(z)}$$

Example 5

- The first-order lowpass transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

can be expressed as

$$H_{LP}(z) = \frac{1}{2}\left(1+\frac{-\alpha+z^{-1}}{1-\alpha z^{-1}}\right) = \frac{1}{2}[A_0(z)+A_1(z)]$$

where

$$A_0(z) = 1, \qquad A_1(z) = \frac{-\alpha+z^{-1}}{1-\alpha z^{-1}}$$

Example 5

- Its power-complementary highpass transfer function is thus given by

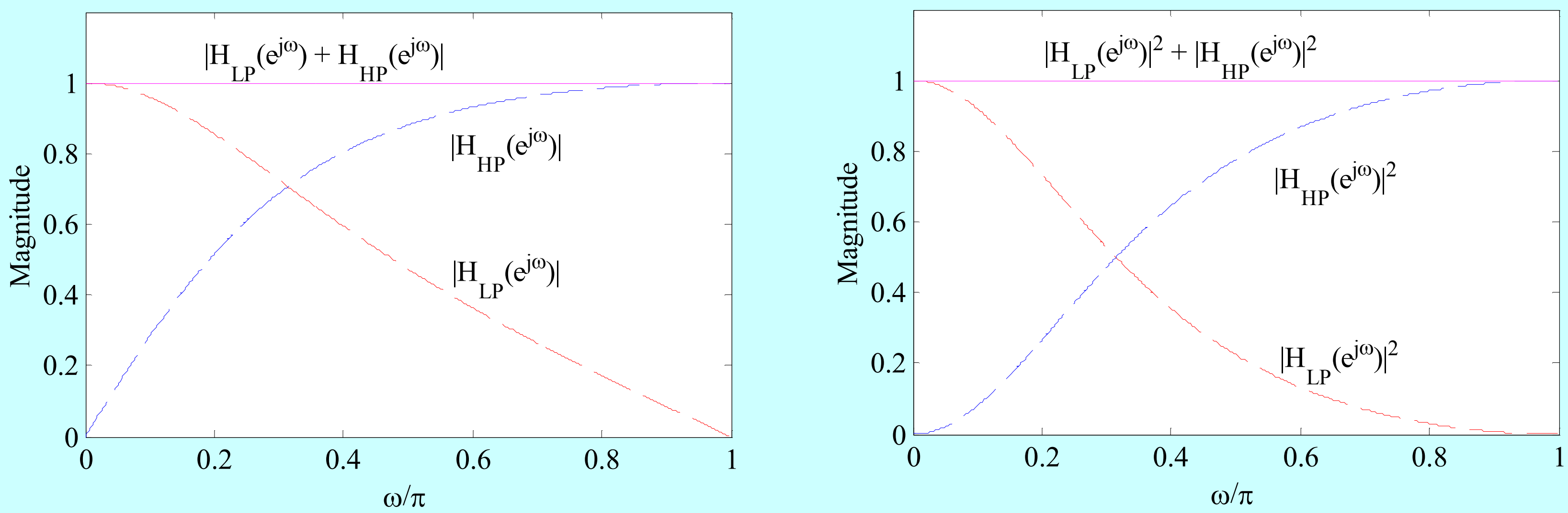
$$H_{HP}(z) = \frac{1}{2}[A_0(z) - A_1(z)] = \frac{1}{2}\left(1 - \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}\right)$$

$$= \frac{1+\alpha}{2}\left(\frac{1 - z^{-1}}{1 - \alpha z^{-1}}\right)$$

- The above expression is precisely the first-order highpass transfer function described earlier

Complementary Transfer Functions

- Figure below demonstrates the allpass complementary property and the power complementary property of $H_{LP}(z)$ and $H_{HP}(z)$



Complementary Transfer Functions

Power-Symmetric Filters

- A real-coefficient causal digital filter with a transfer function $H(z)$ is said to be a **power-symmetric filter** if it satisfies the condition

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K$$

where $K > 0$ is a constant

Complementary Transfer Functions

- It can be shown that the gain function G(ω) of a power-symmetric transfer function at ω = π is given by

$$10\log_{10} K - 3\ dB$$

- If we define $G(z) = H(-z)$, then it follows from the definition of the power-symmetric filter that $H(z)$ and $G(z)$ are power-complementary as

$$H(z)H(z^{-1}) + G(z)G(z^{-1}) = \text{a constant}$$

Complementary Transfer Functions

Conjugate Quadratic Filter

- If a power-symmetric filter has an FIR transfer function $H(z)$ of order N, then the FIR digital filter with a transfer function

$$G(z) = z^{-1} H(z^{-1})$$

is called a **conjugate quadratic filter** of $H(z)$ and vice-versa

Complementary Transfer Functions

- It follows from the definition that $G(z)$ is also a power-symmetric causal filter
- It also can be seen that a pair of conjugate quadratic filters $H(z)$ and $G(z)$ are also power-complementary

Example 6

- Let $H(z) = 1 - 2z^{-1} + 6z^{-2} + 3z^{-3}$
- We form

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1})$$

$$= (1 - 2z^{-1} + 6z^{-2} + 3z^{-3})(1 - 2z + 6z^{2} + 3z^{3})$$

$$+ (1 + 2z^{-1} + 6z^{-2} - 3z^{-3})(1 + 2z + 6z^{2} - 3z^{3})$$

$$= (3z^{3} + 4z + 50 + 4z^{-1} + 3z^{-3})$$

$$+ (-3z^{3} - 4z + 50 - 4z^{-1} - 3z^{-3}) = 100$$

- ⇒ $H(z)$ is a power-symmetric transfer function

Digital Two-Pairs

- The LTI discrete-time systems considered so far are single-input, single-output structures characterized by a transfer function
- Often, such a system can be efficiently realized by interconnecting two-input, two-output structures, more commonly called **two-pairs**

Digital Two-Pairs

- Figures below show two commonly used block diagram representations of a two-pair

- Here Y_1 and Y_2 denote the two outputs, and X_1 and X_2 denote the two inputs, where the dependencies on the variable z have been omitted for simplicity

Digital Two-Pairs

- The input-output relation of a digital two-pair is given by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

- In the above relation the matrix τ given by

$$\tau = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

is called the **transfer matrix** of the two-pair

Digital Two-Pairs

- It follows from the input-output relation that the transfer parameters can be found as follows:

$$t_{11} = \left.\frac{Y_1}{X_1}\right|_{X_2=0}, \qquad t_{12} = \left.\frac{Y_1}{X_2}\right|_{X_1=0}$$

$$t_{21} = \left.\frac{Y_2}{X_1}\right|_{X_2=0}, \qquad t_{22} = \left.\frac{Y_2}{X_2}\right|_{X_1=0}$$

Digital Two-Pairs

- An alternative characterization of the two-pair is in terms of its chain parameters as

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Y_2 \\ X_2 \end{bmatrix}$$

where the matrix Γ given by

$$\Gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is called the **chain matrix** of the two-pair

Digital Two-Pairs

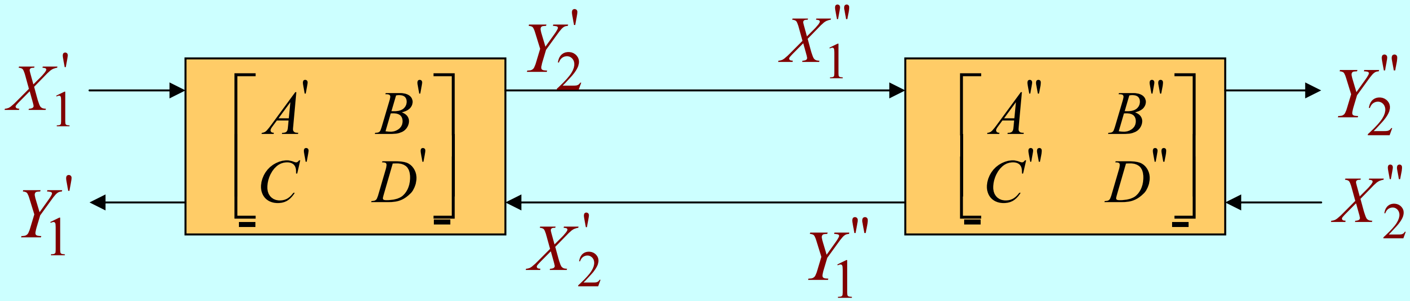
- The relation between the transfer parameters and the chain parameters are given by

$$t_{11} = \frac{C}{A}, \quad t_{12} = \frac{AD - BC}{A}, \quad t_{21} = \frac{1}{A}, \quad t_{22} = -\frac{B}{A}$$

$$A = \frac{1}{t_{21}}, \quad B = -\frac{t_{22}}{t_{21}}, \quad C = \frac{t_{11}}{t_{21}}, \quad D = \frac{t_{12}t_{21} - t_{11}t_{22}}{t_{21}}$$

Two-Pair Interconnection Schemes

Cascade Connection - Γ-cascade



- Here

$$\begin{bmatrix} X_1' \\ Y_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} Y_2' \\ X_2' \end{bmatrix}$$

$$\begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix} = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{bmatrix} Y_2'' \\ X_2'' \end{bmatrix}$$

Two-Pair Interconnection Schemes

- But from figure, $X_1'' = Y_2'$ and $Y_1'' = X_2'$
- Substituting the above relations in the first equation on the previous slide and combining the two equations we get

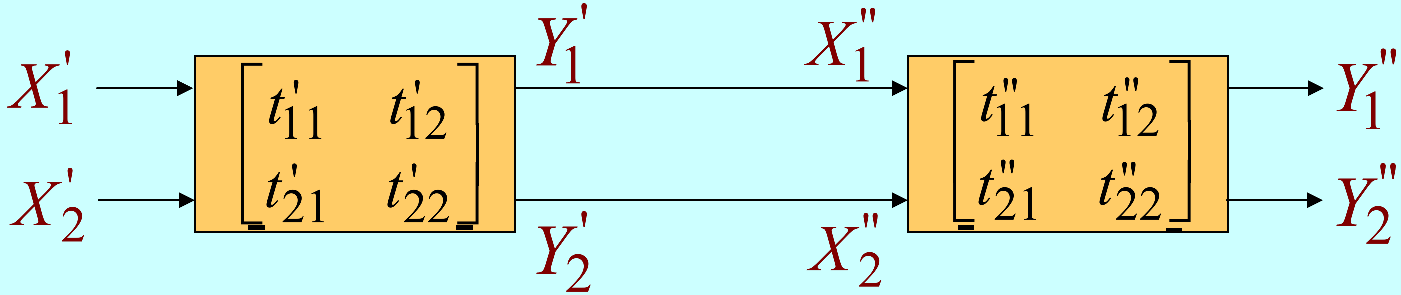
$$\begin{bmatrix} X_1' \\ Y_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{bmatrix} Y_2'' \\ X_2'' \end{bmatrix}$$

- Hence,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}$$

Two-Pair Interconnection Schemes

Cascade Connection - τ-cascade



- Here $\begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} = \begin{bmatrix} t''_{11} & t''_{12} \\ t''_{21} & t''_{22} \end{bmatrix} \begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix}$

Two-Pair Interconnection Schemes

- But from figure, $X_1'' = Y_1'$ and $X_2'' = Y_2'$
- Substituting the above relations in the first equation on the previous slide and combining the two equations we get

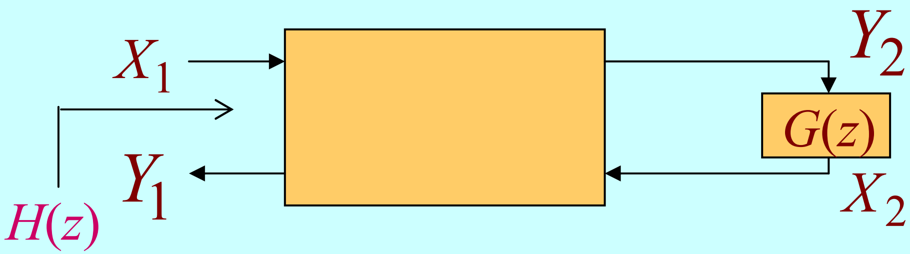
$$\begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} = \begin{bmatrix} t_{11}'' & t_{12}'' \\ t_{21}'' & t_{22}'' \end{bmatrix} \begin{bmatrix} t_{11}' & t_{12}' \\ t_{21}' & t_{22}' \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$$

- Hence,

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11}'' & t_{12}'' \\ t_{21}'' & t_{22}'' \end{bmatrix} \begin{bmatrix} t_{11}' & t_{12}' \\ t_{21}' & t_{22}' \end{bmatrix}$$

Two-Pair Interconnection Schemes

Constrained Two-Pair



- It can be shown that

$$H(z) = \frac{Y_1}{X_1} = \frac{C + D \cdot G(z)}{A + B \cdot G(z)}$$

$$= t_{11} + \frac{t_{12} t_{21} G(z)}{1 - t_{22} G(z)}$$