E2.5 Signals & Linear Systems

Tutorial Sheet 6 – Solutions

1. a)

$$F(\omega) = \int_0^T e^{-at} e^{-j\omega t}\,dt$$
$$= \int_0^T e^{-(a+j\omega)t}\,dt$$
$$= \frac{1 - e^{-(a+j\omega)T}}{a + j\omega}$$

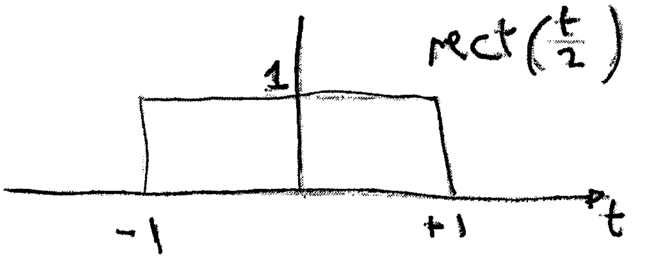
//

b)

$$F(\omega) = \int_0^T e^{at} e^{-j\omega t}\,dt$$
$$= \int_0^T e^{-(-a+j\omega)t}\,dt$$
$$= \frac{1 - e^{-(-a+j\omega)T}}{-a + j\omega}$$

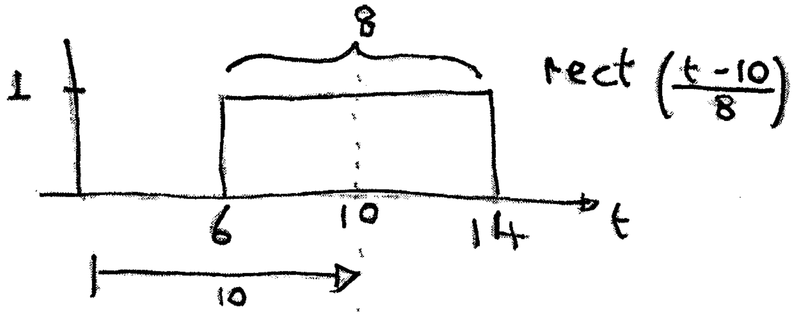
//

2 a)



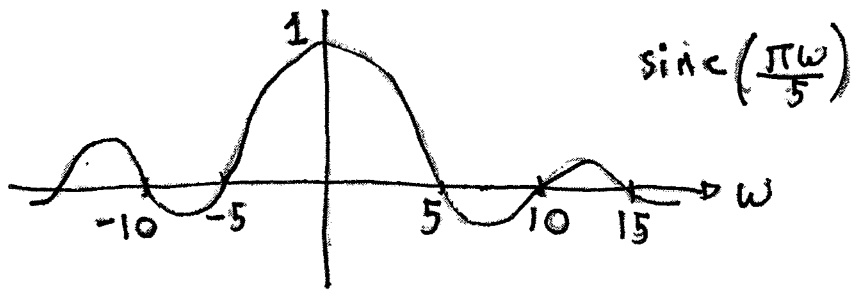
rect(t/2)
1
-1
+1
t

b)



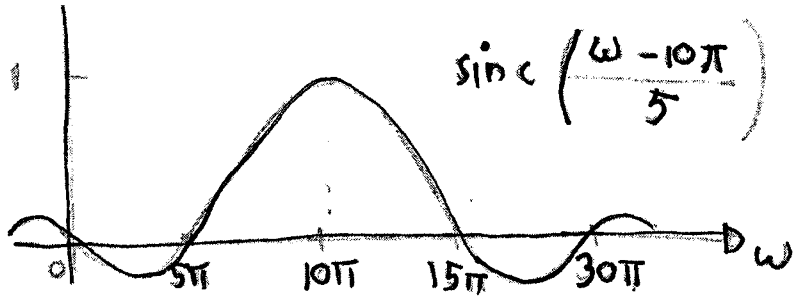
8
rect((t-10)/8)
1
6
10
14
t
10

c)



1
sinc(πω/5)
-10
-5
5
10
15
ω

d)



1
sinc((ω-10π)/5)
0
5π
10π
15π
30π
ω

3. a) From lec 10, slide 14. FT table
pair # 10,

$$\underbrace{u(t)}_{f(t)} \Longleftrightarrow \underbrace{\pi\delta(\omega) + \frac{1}{j\omega}}_{F(\omega)}$$

Duality property is given in lect 10, slide 5.

Applying the duality property yields,

$$\underbrace{\pi\delta(t) + \frac{1}{jt}}_{F(t)} \Longleftrightarrow \underbrace{2\pi\, u(-\omega)}_{2\pi f(-\omega)}$$

$$\therefore \frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \Longleftrightarrow u(-\omega)$$

Apply scaling property ($a = -1$),

$$\frac{1}{2}\left[\delta(-t) - \frac{1}{j\pi t}\right] \Longleftrightarrow u(\omega)$$

$\delta(-t) = \delta(t)$

$$\therefore \frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right] \Longleftrightarrow u(\omega)$$

//

3 . b) We need to show

$$\frac{1}{t} \Longleftrightarrow -j\pi \,\mathrm{sgn}(\omega)$$

(Error in question: "sng" should be "sgn")

From FT table, Lec 10/14, #12

$$\underbrace{\mathrm{sgn}(t)}_{f(t)} \Longleftrightarrow \underbrace{\frac{2}{j\omega}}_{F(\omega)}$$

∴ Using duality property:

$$\underbrace{\frac{2}{jt}}_{F(t)} \Longleftrightarrow \underbrace{2\pi \,\mathrm{sgn}(-\omega)}_{2\pi f(-\omega)} = -2\pi \,\mathrm{sgn}(\omega)$$

$$\therefore \quad \frac{1}{t} \Longleftrightarrow -j\pi \,\mathrm{sgn}(\omega) \quad //$$

c) show $\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j\sin(T\omega)$

From FT table, #10

$$\underbrace{\sin \omega_0 t}_{f(t)} \Longleftrightarrow j\pi \underbrace{\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]}_{F(\omega)}$$

$$\therefore \quad j\pi\left[\delta(t+\omega_0) - \delta(t-\omega_0)\right] \Longleftrightarrow 2\pi \sin(-\omega_0\omega)$$

$$= -2\pi \sin(\omega_0 \omega)$$

Let $\omega_0 = T$ yields

$$j\pi\left[\delta(t+T) - \delta(t-T)\right] \Longleftrightarrow -2\pi \sin(\omega T)$$

$$\therefore \quad \delta(t+T) - \delta(t-T) \Longleftrightarrow 2j \sin(\omega T) \quad //$$

4. Fig (b) $f_1(t) = f(-t)$

$\therefore F_1(\omega) = F(-\omega) = \frac{1}{\omega^2}\left[e^{-j\omega} + j\omega e^{-j\omega} - 1\right]$ //

Fig (c) $f_2(t) = f(t-1) + f_1(t-1)$.

$$\therefore F_2(\omega) = \left[F(\omega)e^{-j\omega} + F_1(\omega)e^{-j\omega}\right].$$

$$= \left[F(\omega) + F(-\omega)\right]e^{-j\omega}$$

$$= \frac{2}{\omega^2}\left(\cos\omega + \omega\sin\omega - 1\right)$$ //

Fig (d) $f_3(t) = f(t-1) + f_1(t+1)$.

$$\therefore F_3(\omega) = F(\omega)e^{-j\omega} + F(-\omega)e^{j\omega}$$

$$= \frac{1}{\omega^2}\left[2 - 2\cos\omega\right]$$

$$= \frac{4}{\omega^2}\sin^2\frac{\omega}{2} = \mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$ //

Fig (e) $f_4(t) = f(t-\frac{1}{2}) + f_1(t+\frac{1}{2})$

$$\therefore F_4(\omega) = e^{-j\omega/2}F(\omega) + e^{j\omega/2}F(-\omega)$$

$$= \frac{e^{-j\omega/2}}{\omega^2}\left[e^{j\omega} - j\omega e^{j\omega} - 1\right] + \frac{e^{j\omega/2}}{\omega^2}\left[e^{-j\omega} + j\omega e^{-j\omega} - 1\right]$$

$$= \frac{1}{\omega^2}\left[2\omega\sin\frac{\omega}{2}\right]$$

$$= \mathrm{sinc}\left(\frac{\omega}{2}\right)$$ //

4) Fig (f) $f_5(t)$ can be constructed in 3 steps.

step 1 : Time expand $f(t)$ by a factor of 2.

$$f\left(\frac{t}{2}\right) \iff 2F(2\omega) = \frac{1}{2\omega^2}\left(e^{-j2\omega} - j2\omega e^{j2\omega} - 1\right)$$

Step 2 : Delay this by 2 seconds.

$$f\left(\frac{t-2}{2}\right) \iff \frac{1}{2\omega^2}\left(e^{-j2\omega} - j2\omega e^{j2\omega} - 1\right)e^{-j2\omega}$$

$$= \frac{1}{2\omega^2}\left(1 - j2\omega - e^{-j2\omega}\right)$$

Step 3: Multiply result by 1.5 :

$$f_5(t) = 1.5\, f\left(\frac{t-2}{2}\right) \iff \frac{3}{4\omega^2}\left(1 - j2\omega - e^{-j2\omega}\right)$$

//

5. (a) $f(t)$ is a triangular pulse $\Delta\left(\frac{t}{2\pi}\right)$ multiplied by $\cos 10t$.

$$f(t) = \Delta\left(\frac{t}{2\pi}\right) \cos 10t .$$

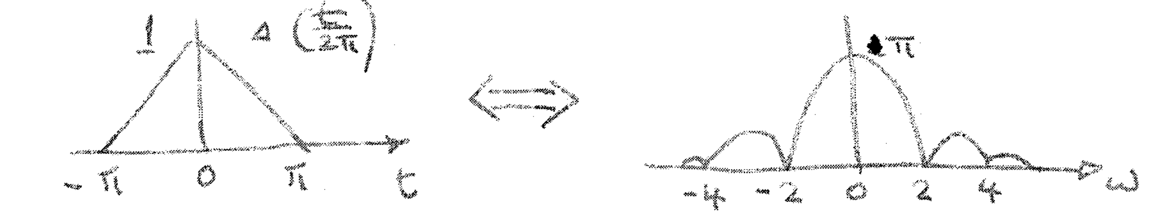
$\therefore$ From FT table pair # 19,

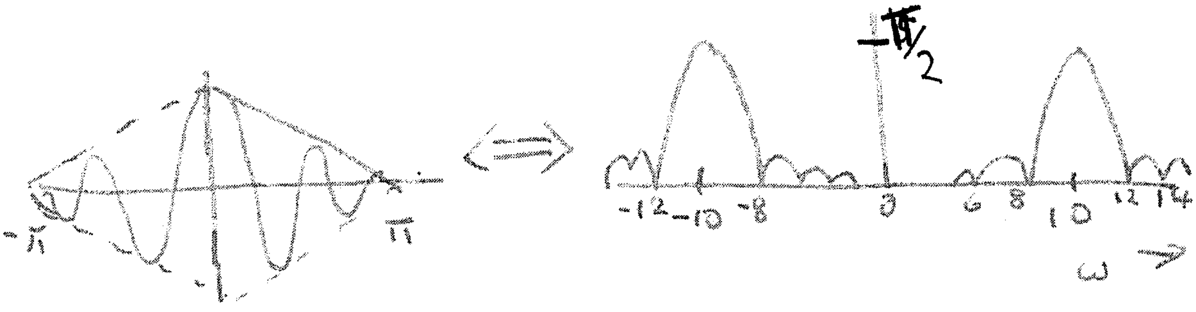
$$\Delta\left(\frac{t}{2\pi}\right) \iff \pi \operatorname{sinc}^2 \frac{\pi\omega}{2} .$$

Using modulation property, it follows:

$$\Delta\left(\frac{t}{2\pi}\right) \cos 10t \iff \frac{\pi}{2}\left\{\operatorname{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \operatorname{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}$$

Since the signal is an even function, the Fourier transform is real, $\therefore$ needs only amplitude spectrum (phase is zero).





5. (b)

$f(t)$ is simply shifted in time by 2π.

$\therefore$ FT is the same as (a), but multiplied by $e^{-j\omega(2\pi)}$.

$$\angle F(\omega) = -2\pi\omega \quad \text{(i.e. linear phase)}.$$

$$F(\omega) = \frac{\pi}{2}\left\{\operatorname{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \operatorname{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}e^{-j2\pi\omega}$$

(c)

$f(t)$ is the same as (b) except the $\Delta\left(\frac{t}{2\pi}\right)$ is replace by $\operatorname{rect}\left(\frac{t}{2\pi}\right)$.

$$\operatorname{rect}\left(\frac{t}{2\pi}\right) \Longleftrightarrow 2\pi \operatorname{sinc}(\pi\omega).$$

$$\therefore F(\omega) = \pi\left\{\operatorname{sinc}\left[\pi(\omega-10)\right] + \operatorname{sinc}\left[\pi(\omega+10)\right]\right\}e^{-j2\pi\omega}$$