E2.5 Signals & Linear Systems

Tutorial Sheet 3 - Solutions

1. (a) $u(t) * u(t)$

$$= \int_0^t u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_0^t d\tau = \tau \Big|_0^t = t \qquad \text{for } t \geq 0$$

$$= 0 \qquad \text{for } t < 0$$

$\therefore\ u(t) * u(t) = t\, u(t)$

(b) $e^{-at} u(t) * e^{-bt} u(t)$

$$= \int_0^t e^{-a\tau} e^{-b(t-\tau)} d\tau = e^{-bt} \int_0^t e^{(b-a)\tau} d\tau$$

$$= \frac{e^{-bt}}{b-a} e^{(b-a)\tau} \Big|_0^t = \frac{e^{-bt}}{b-a} \left[e^{(b-a)t} - 1 \right]$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}.$$

Because both functions are casual, their convolution is zero for $t < 0$.

$\therefore\ y(t) = e^{-at} u(t) * e^{-b(t)} u(t)$

$$= \left(\frac{e^{-at} - e^{-bt}}{b-a} \right) u(t).$$

(c) Both functions are causal, $\therefore$

$$t\,u(t) * u(t) = \int_0^t \tau\, u(\tau)\, u(t-\tau)\, d\tau$$

For the range of integration, $0 \le \tau \le t$,

and $u(\tau)$ at $\tau=0$ is $u(0)=1$, at $\tau=t$ $u(t)=1$

and $u(t-\tau)$ at $\tau=0$ is $u(t)=1$, at $\tau=t$, $u(0)=1$

$\therefore$ $u(\tau) = u(t-\tau) = 1$, and

$$t\,u(t) * u(t) = \int_0^t \tau\, d\tau$$

$$= \frac{t^2}{2} \quad \text{for } t \geq 0$$

$$= 0 \quad \text{for } t < 0$$

$$\therefore\ \underline{y(t) = \frac{1}{2} t^2 u(t)}$$

2. a) $y(t) = \sin t\, u(t) * u(t)$

$$= \left[\int_0^t \sin\tau\, u(\tau)\, u(t-\tau)\, d\tau\right] u(t)$$

$$= \left[\int_0^t \sin\tau\, d\tau\right] u(t) \qquad \begin{matrix} u(\tau)=1 \\ u(t-\tau)=1 \end{matrix}$$

$$= (1-\cos t)\, u(t).$$

b) Similarily,

$$y(t) = \left[\int_0^t \cos\tau\, d\tau\right] u(t) = \sin t\, u(t)$$

3. a) $h(t) = e^{-t} u(t)$, $f(t) = u(t)$

$y(t) = e^{-t} u(t) * u(t)$ (Pair #2)

$= (1 - e^{-t})\, u(t)$

b) $f(t) = e^{-2t} u(t)$

$y(t) = e^{-2t} u(t) * e^{-t} u(t)$ (Pair #4)

$= (e^{-t} - e^{-2t})\, u(t)$

c) $f(t) = \sin 3t\; u(t)$

$y(t) = \sin 3t\; u(t) * u^{-t} u(t)$

(Pair #12)

$\alpha = 0,\ \beta = 3,\ \theta = -90^\circ,\ \lambda = -1.$

$\therefore \phi = \tan^{-1}\left[\frac{-3}{-1}\right] = 108.4^\circ$

and $y(t) = \dfrac{\cos(-90^\circ + 108.4^\circ)\, e^{-t} - \cos(3t + 18.4^\circ)}{\sqrt{(0-1)^2 + 3^2}}$

$= \dfrac{0.9485\, e^{-t} - \cos(3t - 18.4^\circ)}{\sqrt{10}}\; u(t)$

— • —— • — Pair #12 —— • —— • —

$e^{-\alpha t} \cos(\beta t + \theta)\, u(t) * e^{\lambda t} u(t)$

$\Longrightarrow \dfrac{\cos(\theta - \phi)\, e^{\lambda t} - e^{-\alpha t} \cos(\beta t + \theta - \phi)}{\sqrt{(\alpha + \lambda)^2 + \beta^2}}\, u(t)$

$\phi = \tan^{-1}\left[-\beta / (\alpha + \lambda)\right]$ §3.3

4. The key is to realise that

$$f(t) = u(t) - u(t-1).$$

$$\therefore \; e^{-t}u(t) * u(t) \Longleftrightarrow \underbrace{(1-e^{-t})\,u(t)}_{z(t)} \quad A$$

$$e^{-t}u(t) * u(t-1) \Longrightarrow z(t-1) \quad B$$

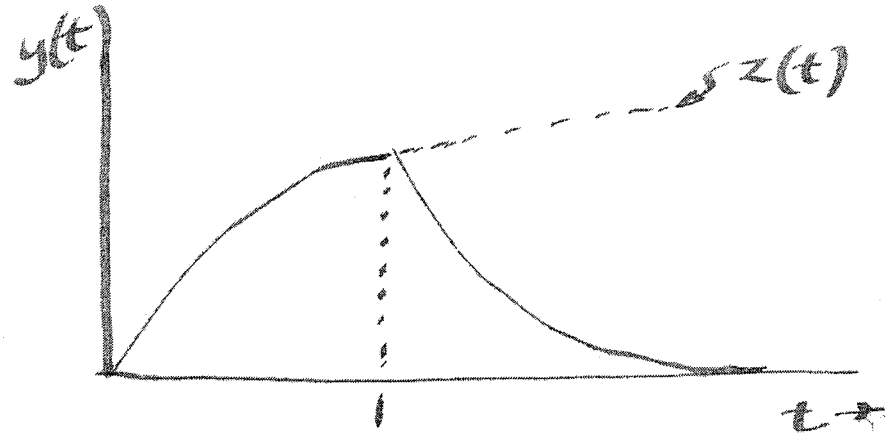
$$= \left[1 - e^{-(t-1)}\right] u(t-1)$$

$$\therefore \; y(t) = e^{-t}u(t) * \left[\, u(t) - u(t-1) \,\right]$$

$$= (\; A - B \;)$$

superposition

$$= (1-e^{-t})\,u(t) - \left[1-e^{-(t-1)}\right] u(t-1)$$



Q5a) $\cdot\ h(t) = -\delta(t) + 2e^{-t}u(t)$

$x(t) = e^{t}u(-t)$

$$
\begin{aligned}
y(t) &= h(t) * x(t) \\
&= [-\delta(t) + 2e^{-t}u(t)] * e^{t}u(-t) \\
&= -\delta(t) * e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t)
\end{aligned}
$$

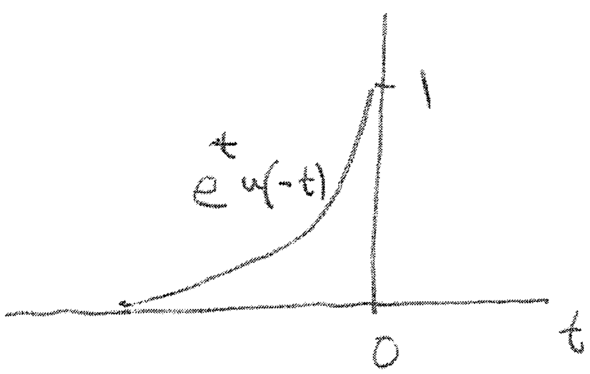
$\boxed{\text{See convolution table no: 13, lecture 5-5}}$

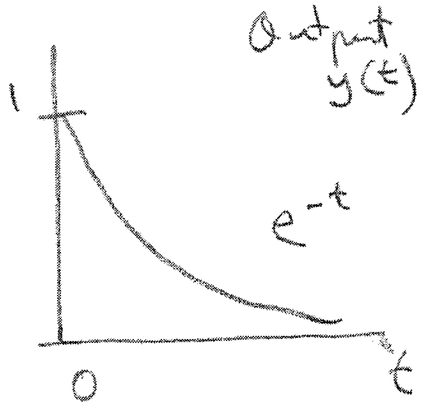
$\Downarrow\ -e^{t}u(-t)$

$\Downarrow\ 2\left[\dfrac{e^{-t}u(t) + e^{t}u(-t)}{2}\right]$

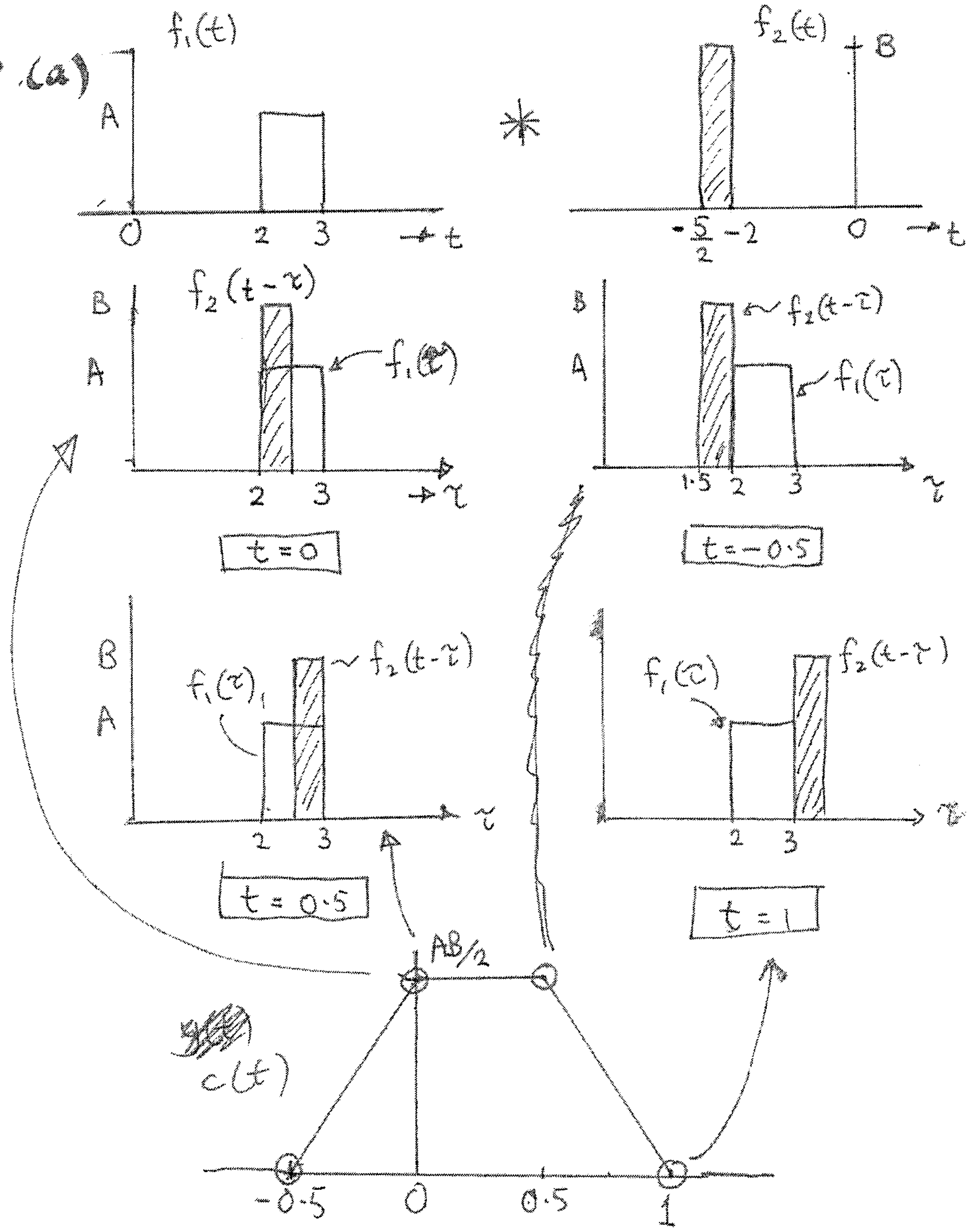
$$
\begin{aligned}
\therefore y(t) &= \cancel{-e^{t}u(-t)} + e^{-t}u(t) + \cancel{e^{t}u(-t)} \\
&= e^{-t}u(t)
\end{aligned}
$$

b) input $x(t)$



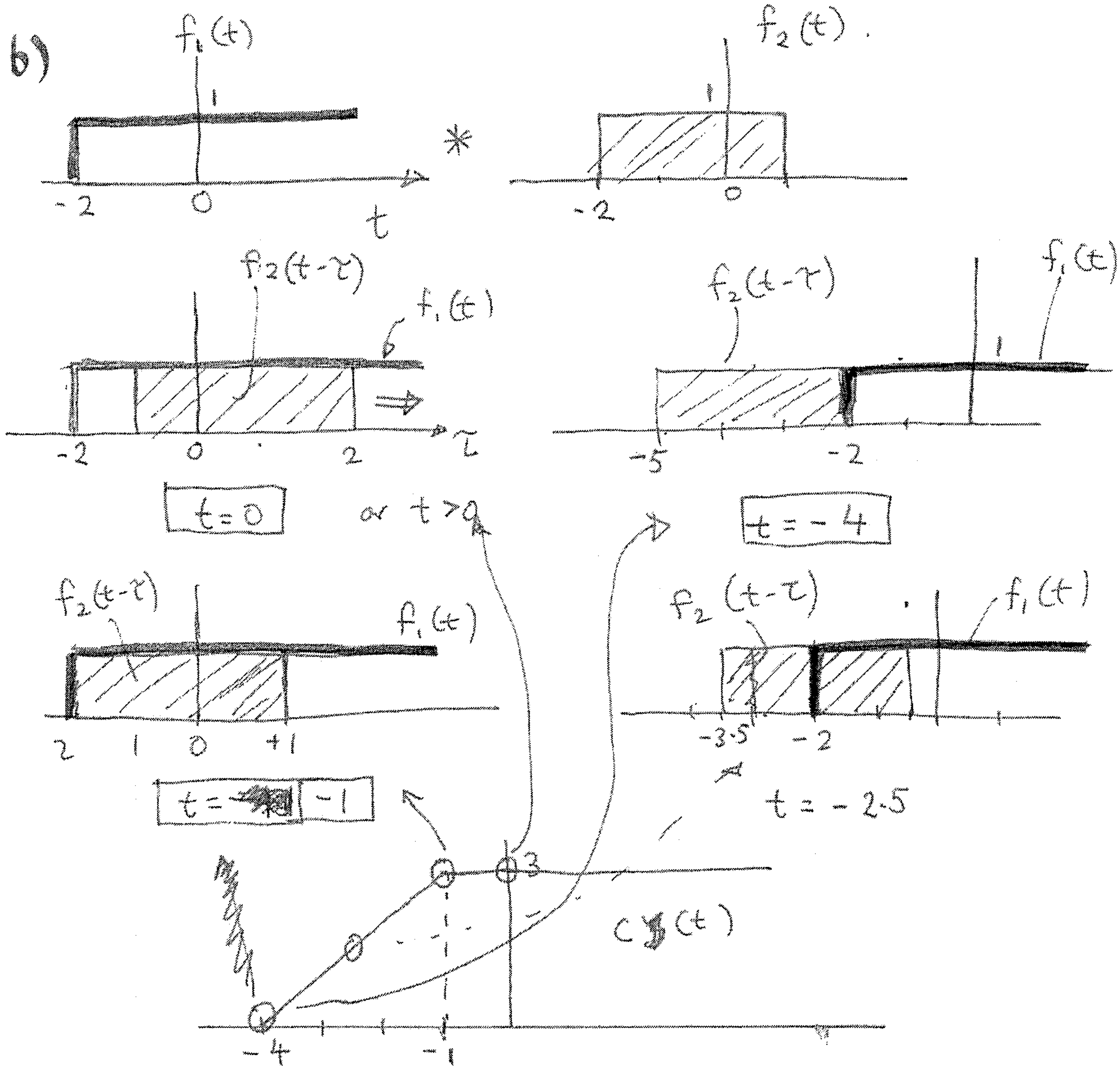


Q6 (a)



f1(t)
A
0
2
3
t
f2(t)
B
-5/2
-2
0
t
f2(t-τ)
B
A
f1(τ)
2
3
τ
t = 0
f2(t-τ)
A
f1(τ)
1.5
2
3
τ
t = -0.5
B
A
f1(τ)
f2(t-τ)
2
3
τ
t = 0.5
f1(τ)
f2(t-τ)
2
3
τ
t = 1
AB/2
c(t)
-0.5
0
0.5
1

6. b)

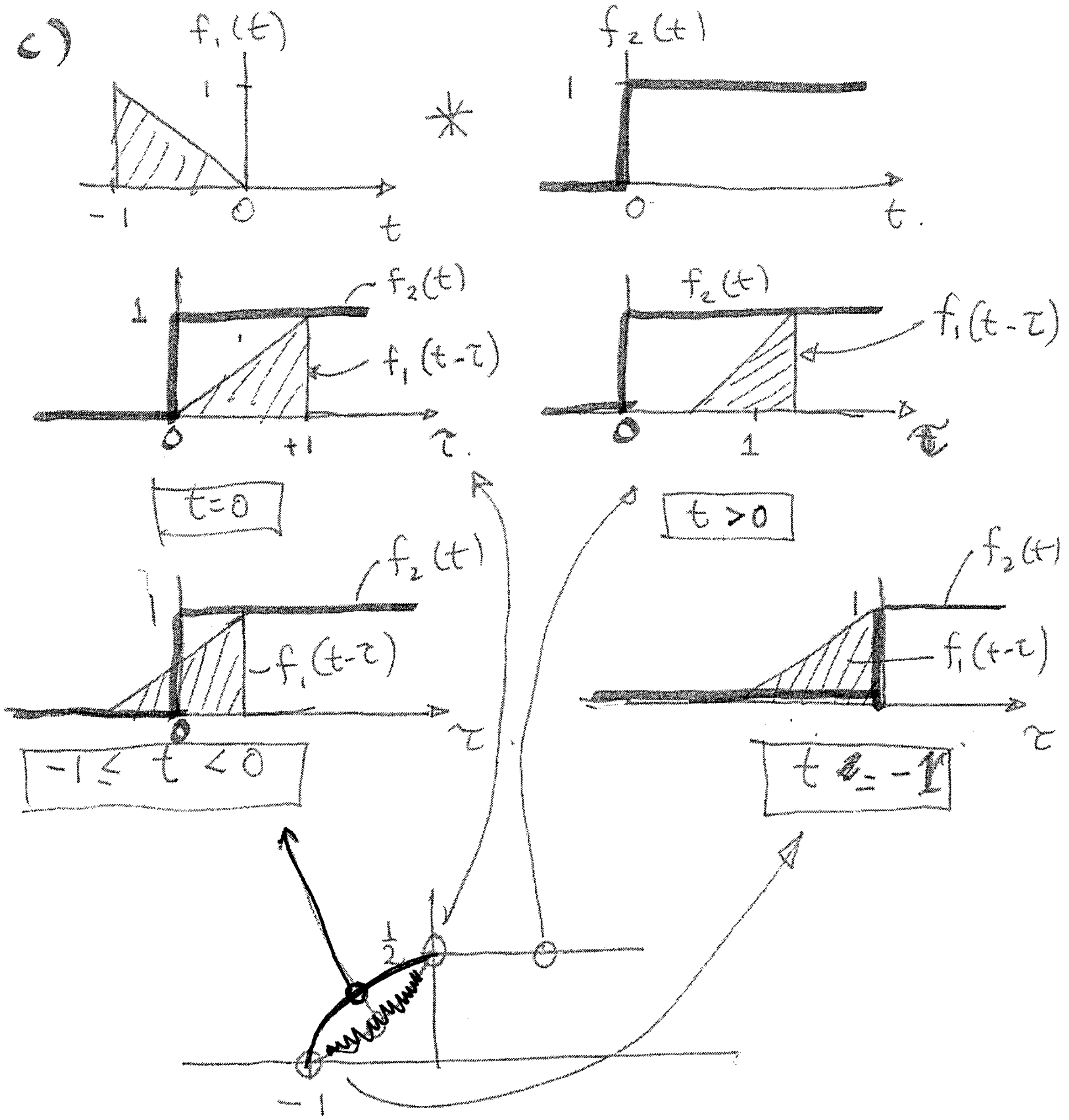


$$c(t) = \int_{-1+t}^{2+t} d\tau = 3 \qquad t \geq -1$$

$$c(t) = \int_{-2}^{2+t} d\tau = t + 4 \qquad -1 \geq t \geq -4$$

$$c(t) = 0 \qquad t \leq -4$$

6 c)



Note: Easy to do $\int f_2(\tau)\, f_1(t-\tau)\, d\tau$

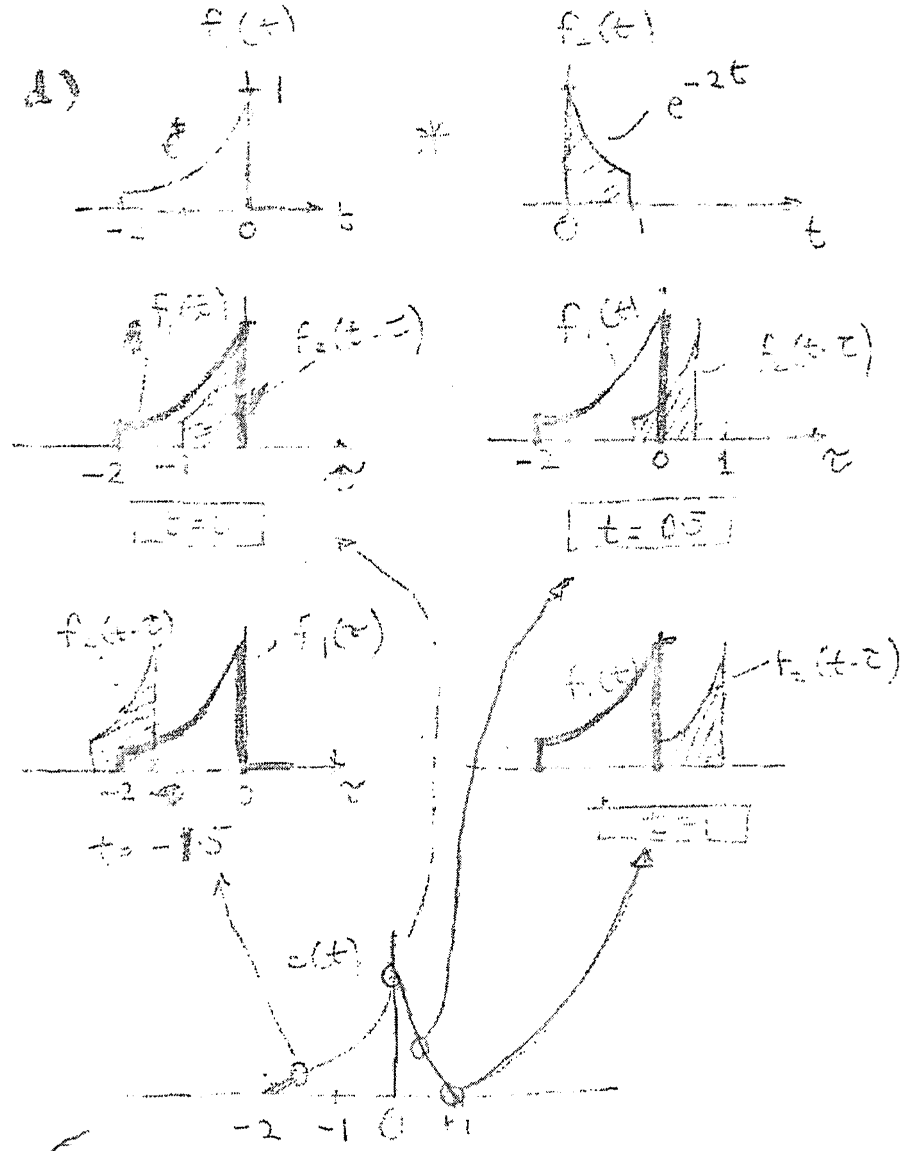
$$c(t) = \int_t^{t+1} (\tau - t)\, d\tau = \frac{1}{2} \qquad t \geq 0$$

$$c(t) = \int_0^{t+1} (\tau - t)\, d\tau = \frac{1}{2}(1 - t^2) \qquad 0 \geq t \geq -1$$

$$c(t) = 0 \qquad t < -1$$

S3·8

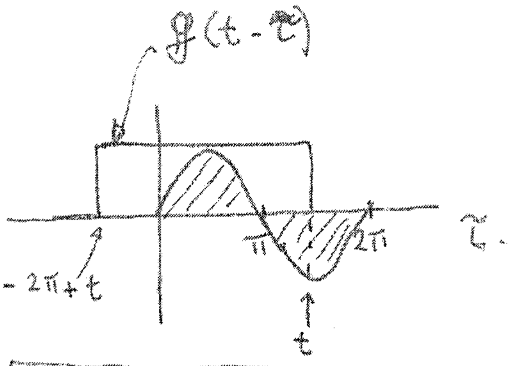
(c 1)



$$f_1(t) * f_2(t) = \begin{cases} 0 & t < -2 \\ \displaystyle\int_{-2}^{t} e^{\tau} e^{-2(t-\tau)} \, d\tau & -2 \le t < -1 \\ \displaystyle\int_{-1+t}^{t} e^{\tau} e^{-2(t-\tau)} \, d\tau & -1 \le t < 0 \\ \displaystyle\int_{-1+t}^{0} e^{\tau} e^{-2(t-\tau)} \, d\tau & 0 \le t < 1 \\ 0 & t > 1 \end{cases}$$

53.9

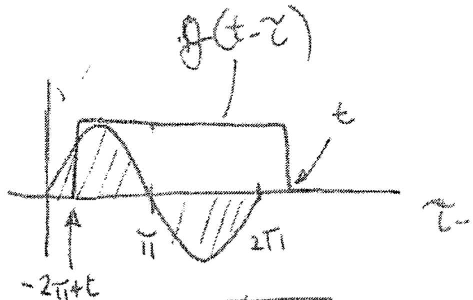
7)



$\boxed{0 \le t \le 2\pi}$

$$f(t) * g(t) = \int_0^t \sin\tau \, d\tau \quad 0 \le t \le 2\pi$$

$$= 1 - \cos t$$



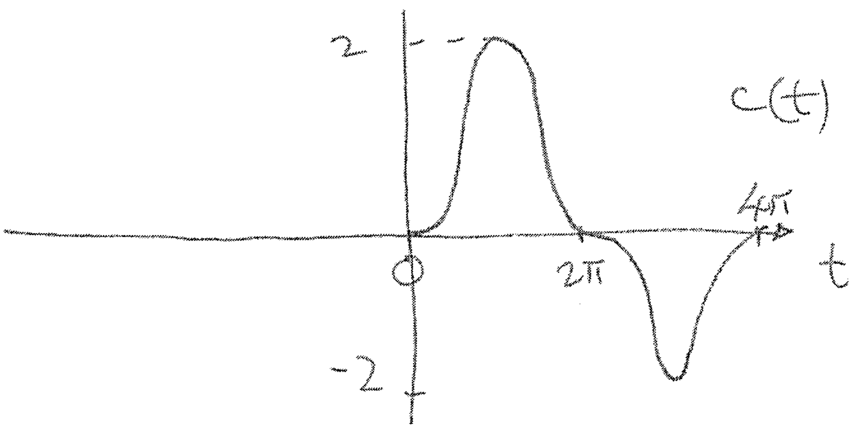
$\cancel{4\pi \ge t > 2\pi}$

$\boxed{2\pi < t \le 4\pi}$

$$f(t) * g(t) = \int_{t-2\pi}^{2\pi} \sin \tau \, d\tau$$

$$2\pi < t \le 4\pi$$

$$= \cos t - 1$$



53·10

~~53·9~~