

E2.5 Signals & Linear Systems.

Tutorial Sheet 2 Zero-input Responses

Solutions

1. System equation is

$$(D^2 + 4D + 4)y(t) = Df(t).$$

Characteristic polynomial is:

$$\lambda^2 + 4\lambda + 4$$

Characteristic equation is:

$$\lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)^2.$$

This has repeated roots of $\lambda = -2$ (twice).

For repeated roots, characteristic modes are e^{-2t} and $t e^{-2t}$.

Therefore, zero-input response $y_0(t)$ is

$$y_0(t) = c_1 e^{-2t} + c_2 t e^{-2t}; \quad y_0(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$$

Letting $t=0$, substituting initial conditions gives:-

$$\left. \begin{array}{l} 3 = c_1 \\ -4 = -2c_1 + c_2 \end{array} \right\} \Rightarrow \begin{array}{l} c_1 = 3 \\ c_2 = 2. \end{array}$$

$$\therefore \underline{\underline{y_0(t) = (3 + 2t)e^{-2t}}}$$

2. System equation is $D(D+1)y(t) = (D+2)f(t)$.

Characteristic polynomial is

$$\lambda(\lambda+1) = \lambda^2 + \lambda$$

Characteristic equation is

$$\lambda(\lambda+1) = 0$$

Characteristic ~~roots~~ roots are 0 and -1 .

Characteristic modes are 1 and e^{-t} .

Therefore

$$y_0(t) = C_1 + C_2 e^{-t}$$

$$\dot{y}_0(t) = -C_2 e^{-t}$$

At $t=0$, $y(0) = 1$, $\dot{y}(0) = 1$.

$$\therefore \begin{cases} 1 = C_1 + C_2 \\ 1 = -C_2 \end{cases} \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = -1 \end{cases}$$

$$\therefore \underline{\underline{y_0(t) = 2 - e^{-t}}}$$

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3. System equation is

$$(D^2 + 9)y(t) = (3D + 2)f(t).$$

Characteristic equations & roots are :-

$$\lambda^2 + 9 = 0, \Rightarrow (\lambda + j3)(\bar{\lambda} - j3) = 0.$$

$$\therefore \lambda = \pm j3$$

Characteristic modes are :-

$$e^{j3t}, e^{-j3t}. \quad (\text{Complex roots})$$

$$\therefore y_0(t) = c \cos(3t + \theta)$$

$$\text{and } \dot{y}_0(t) = -3c \sin(3t + \theta).$$

$$\text{Given at } t=0, y_0(0) = 0, \dot{y}_0(0) = 6,$$

$$\left. \begin{array}{l} 0 = c \cos \theta \\ 6 = -3c \sin \theta \end{array} \right\} \Rightarrow \left. \begin{array}{l} c \cos \theta = 0 \\ c \sin \theta = -2 \end{array} \right\} \Rightarrow \begin{array}{l} c = 2 \\ \theta = -\frac{\pi}{2} \end{array}.$$

Therefore

$$y_0(t) = 2 \cos\left(3t - \frac{\pi}{2}\right)$$

$$= \underline{\underline{2 \sin 3t}}$$

4. a) Remember $\delta(x)$ is located at $x=0$,
and $\delta(t-\tau)$ is located at $\tau=t$, etc.

Therefore ~~the~~ impulse is at $\tau=t$,
and $f(\tau)$ at $\tau=t$ is $f(t)$.

$$\text{Therefore } \int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) d\tau = f(t).$$

b) ~~Similarly~~ Similarly $\delta(\tau)$ is at $\tau=0$,
 $f(t-\tau)$ at $\tau=0$ is $f(t)$.

$$\therefore \int_{-\infty}^{\infty} \delta(\tau) f(t-\tau) d\tau = f(t).$$

$$\text{c) } \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{j\omega 0} = 1.$$

$$\text{d) } \int_{-\infty}^{\infty} \delta(t-2) \sin \pi t dt$$
$$= \sin 2\pi = \emptyset.$$

5. The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0.$$

Characteristic modes are e^{-t} and e^{-3t}

Therefore

$$y_0(t) = c_1 e^{-t} + c_2 e^{-3t}$$

$$\dot{y}_0(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$$

At $t=0$, $y(0)=0$, $\dot{y}(0)=1$ (See notes),

we get:

$$\begin{cases} 0 = c_1 + c_2 \\ 1 = -c_1 - 3c_2 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{cases}$$

Therefore

$$y_0(t) = \frac{1}{2}(e^{-t} - e^{-3t})$$

$$h(t) = [P(D) y_0(t)] u(t)$$

$$= [(D+5) y_0(t)] u(t)$$

$$= [y_0(t) + 5 y_0(t)] u(t)$$

$$= (2e^{-t} - e^{-3t}) u(t)$$

Also check that order of $Q(D) = 2$ (i.e. N)

and order of $P(D) = 1$, $\therefore N > M$.

6. The system equation is more complex here:

$$(D+1)(D^2+5D+6)y(t) = (5D+9)f(t).$$

Characteristic equation:

~~$$(\lambda+1)(\lambda^2+5\lambda+6) = 0$$~~

$$\Rightarrow (\lambda+1)(\lambda+2)(\lambda+3) = 0.$$

Therefore

$$y_0(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

$$\dot{y}_0(t) = -c_1 e^{-t} - 2c_2 e^{-2t} - 3c_3 e^{-3t}$$

$$\ddot{y}_0(t) = c_1 e^{-t} + 4c_2 e^{-2t} + 9c_3 e^{-3t}$$

Since $N=3 > M=1$, at $t=0$, $y_0(0) = \dot{y}_0(0) = 0$,
 $\ddot{y}_0(0) = 1$.

\therefore we obtain:

$$0 = c_1 + c_2 + c_3 \quad c_1 = \frac{1}{2}$$

$$0 = -c_1 - 2c_2 - 3c_3 \Rightarrow c_2 = -1$$

$$1 = c_1 + 4c_2 + 9c_3 \quad c_3 = \frac{1}{2}$$

$$\therefore y_0(t) = \frac{1}{2} e^{-t} - e^{-2t} + \frac{1}{2} e^{-3t}$$

Finally,

$$h(t) = [P(D) y_0(t)] u(t)$$

$$= [5\dot{y}_0(t) + 9y_0(t)] u(t)$$

$$= (2e^{-t} + e^{-2t} - 3e^{-3t}) u(t)$$

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