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Signals and Systems

Lecture 8

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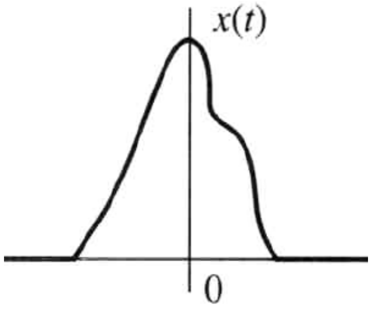
Definition of Fourier transform

- The forward and inverse Fourier transform are defined for aperiodic signals as:

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

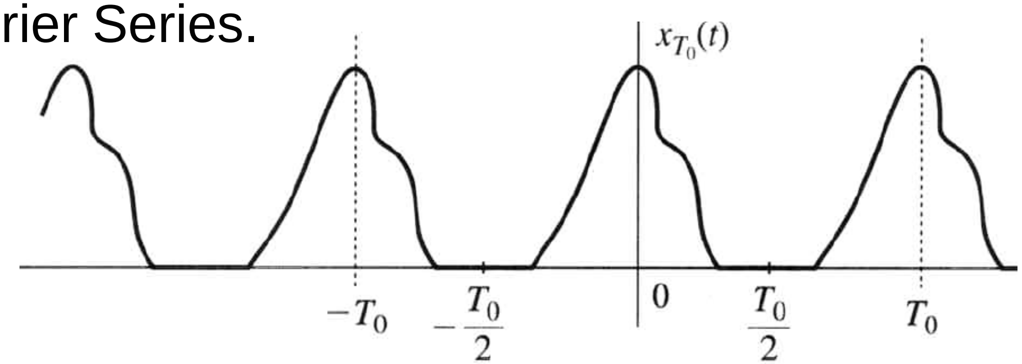


- **You can immediately observe the functional similarity with Laplace transform.**
- For periodic signals we use Fourier Series.

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-jn\omega_0 t}dt \text{ or}$$

$$D_n = \frac{1}{T_0}\int_{\text{one full period}} x_{T_0}(t)e^{-jn\omega_0 t}dt \,, \omega_0 = \frac{2\pi}{T_0}$$



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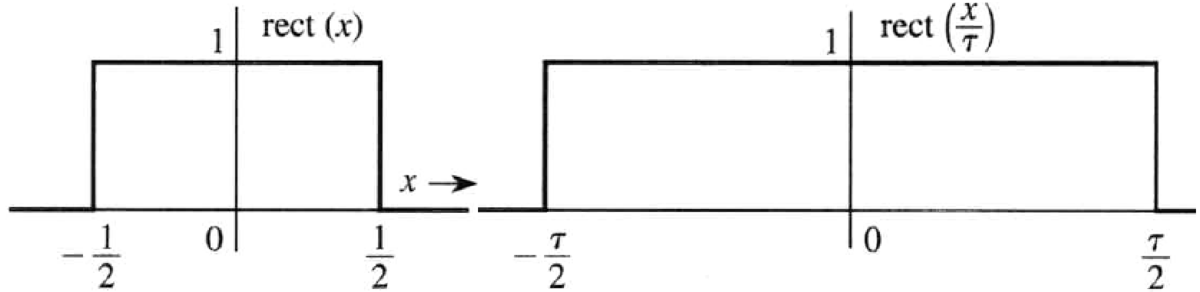
Define three useful functions

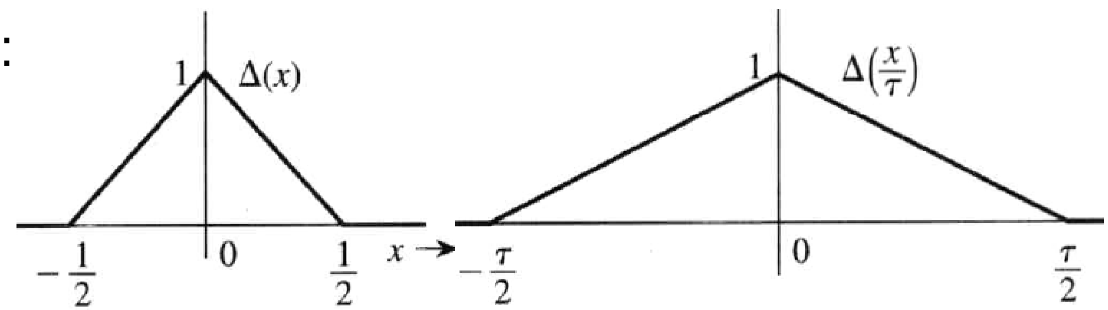
- A unit rectangular window (also called a unit gate) function $\mathrm{rect}(x)$:

$$\mathrm{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$



- A unit triangle function $\Delta(x)$:

$$\Delta(x) = \begin{cases} 0 & |x| \geq \frac{1}{2} \\ 1 - 2|x| & |x| < \frac{1}{2} \end{cases}$$

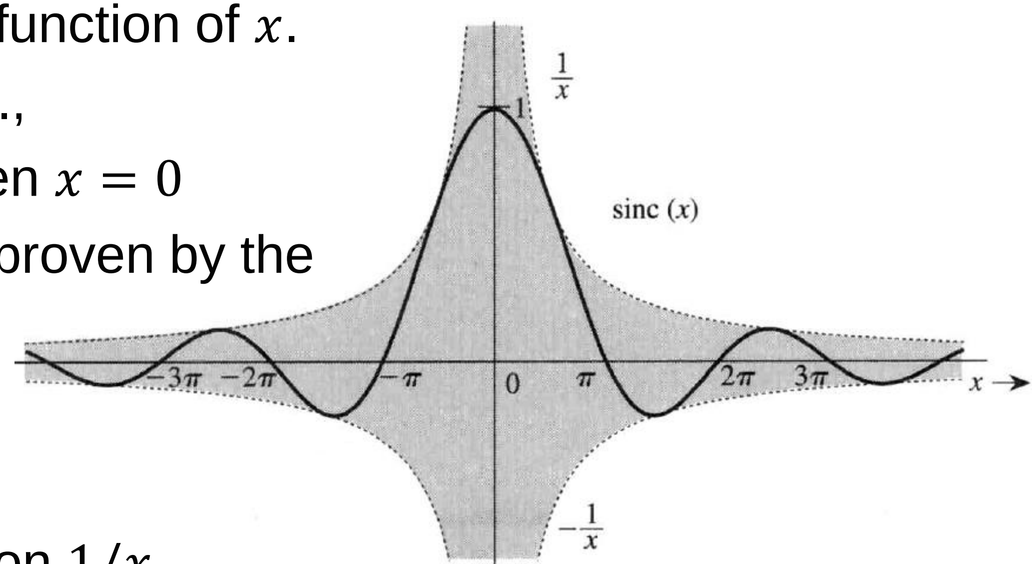


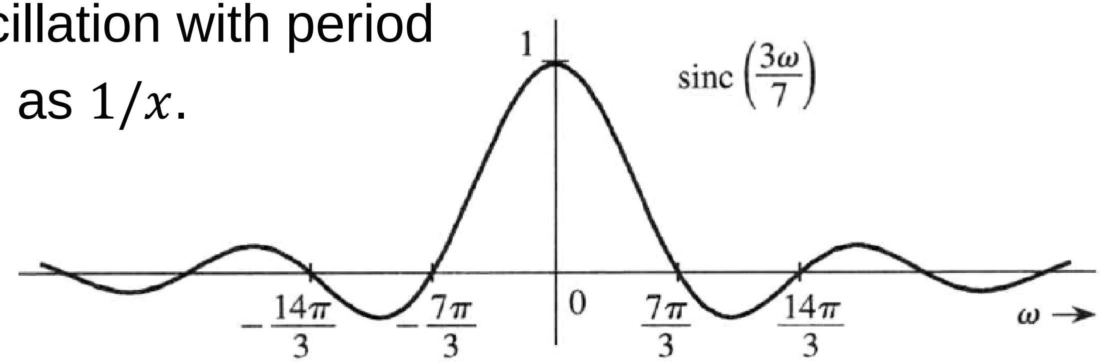
- Interpolation function $\mathrm{sinc}(x)$:

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x} \text{ or } \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

More about the $\mathrm{sinc}(x)$ function

- The $\mathrm{sinc}(x)$ function is an even function of x.
- $\mathrm{sinc}(x) = 0$ when $\sin(x) = 0$, i.e., $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ except when $x = 0$ where $\mathrm{sinc}(0) = 1$. This can be proven by the L'Hospital's rule.
- $\mathrm{sinc}(x)$ is the product of an oscillating signal $\sin(x)$ and a monotonically decreasing function $1/x$. Therefore, it is a damping oscillation with period 2π with amplitude decreasing as $1/x$.





Fourier transform of $x(t) = \mathrm{rect}(t/\tau)$

- Evaluation:

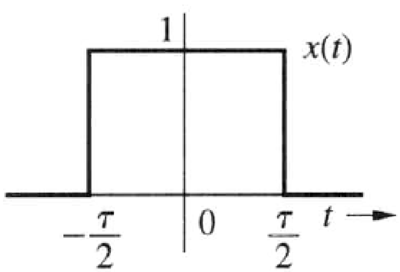
$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

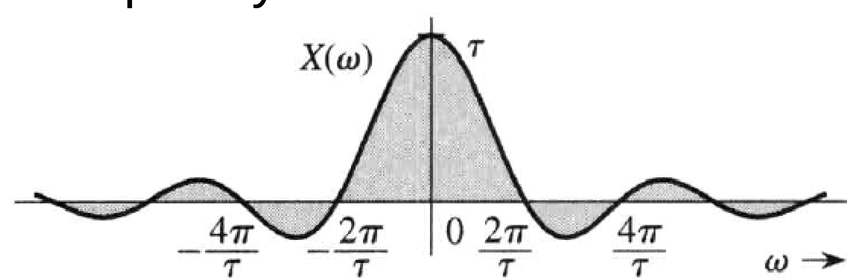
- Since $\mathrm{rect}\left(\frac{t}{\tau}\right) = 1$ for $\frac{-\tau}{2} < t < \frac{\tau}{2}$ and 0 otherwise we have:

$$X(\omega) = \mathcal{F}[x(t)] = \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t} dt = -\frac{1}{j\omega}\left(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right) = \frac{2\sin(\frac{\omega\tau}{2})}{\omega}$$

$$= \tau\,\frac{\sin(\frac{\omega\tau}{2})}{(\frac{\omega\tau}{2})} = \tau \mathrm{sinc}(\frac{\omega\tau}{2}) \Rightarrow \mathcal{F}\left[\mathrm{rect}\left(\frac{t}{\tau}\right)\right] = \tau \mathrm{sinc}(\frac{\omega\tau}{2}) \text{ or } \boxed{\mathrm{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau \mathrm{sinc}(\frac{\omega\tau}{2})}$$

- The bandwidth of the function $\mathrm{rect}\left(\frac{t}{\tau}\right)$ is approximately $\frac{2\pi}{\tau}$.
- Observe that the wider(narrower) the pulse in time the narrower(wider) the lobes of the sinc function in frequency.





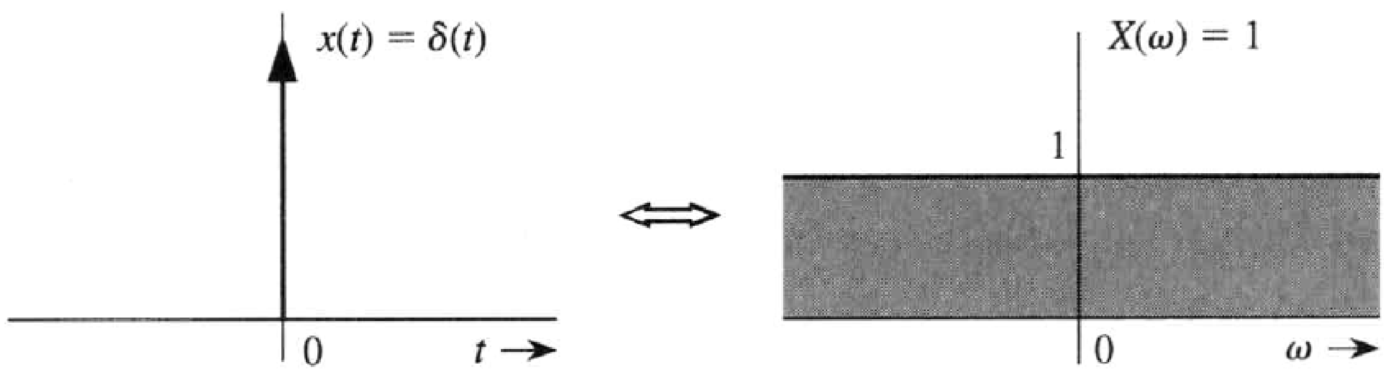
Fourier transform of the unit impulse $x(t) = \delta(t)$

- Using the sampling property of the impulse we get:

$$X(\omega) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

- As we see the unit impulse contains all frequencies (or, alternatively, we can say that the unit impulse contains a component at every frequency.)

$$\delta(t) \Leftrightarrow 1$$



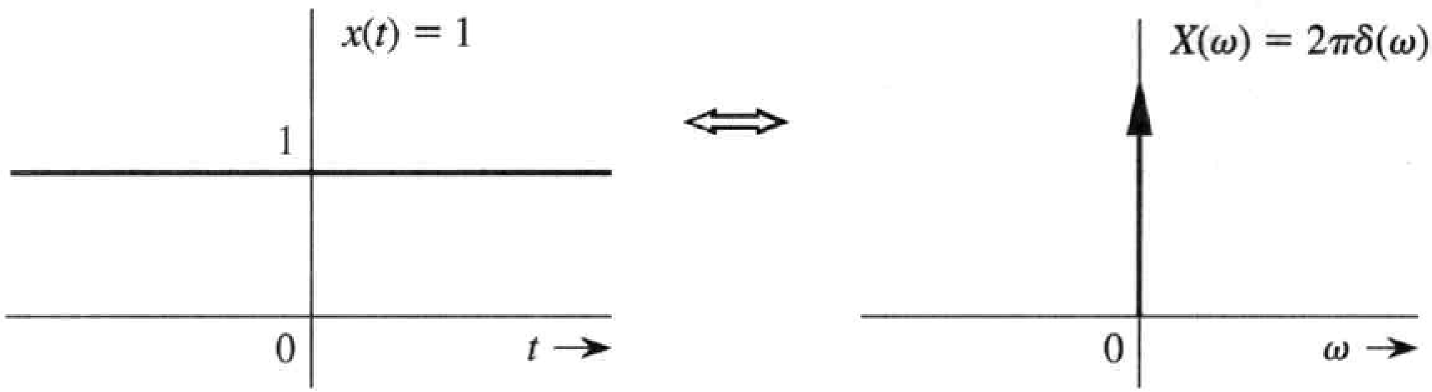
Inverse Fourier transform of $\delta(\omega)$

- Using the sampling property of the impulse we get:

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}$$

- Therefore, the spectrum of a constant signal $x(t) = 1$ is an impulse $2\pi\delta(\omega)$.

$$\frac{1}{2\pi} \Leftrightarrow \delta(\omega) \text{ or } 1 \Leftrightarrow 2\pi\delta(\omega)$$



- By looking at current and previous slide, observe the relationship: wide (narrow) in time, narrow (wide) in frequency.
  - **Extreme case is a constant everlasting function in one domain and a Dirac in the other domain.**

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Inverse Fourier transform of $\delta(\omega - \omega_0)$

- Using the sampling property of the impulse we get:

$$\mathcal{F}^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

- The spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse located at $\omega = \omega_0$.

$$\frac{1}{2\pi}e^{j\omega_0 t} \Leftrightarrow \delta(\omega-\omega_0)$$

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$$

$$e^{-j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega+\omega_0)$$

Fourier transform of an everlasting sinusoid $\cos\omega_0 t$

- Remember the Euler's formula:

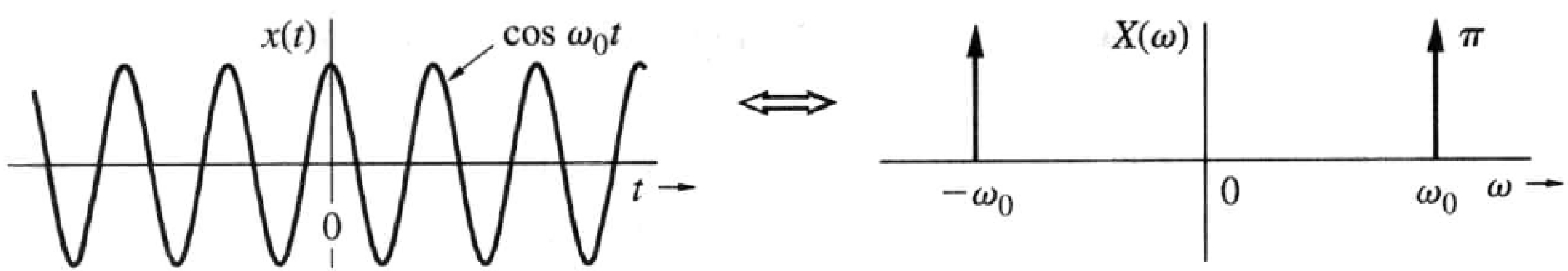
$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\mathcal{F}\{\cos\omega_0 t\} = \mathcal{F}\left\{\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right\} = \frac{1}{2}\,\mathcal{F}\{e^{j\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{e^{-j\omega_0 t}\}$$

- Using the results from previous slides we get:

$$\cos\omega_0 t \Leftrightarrow \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

- The spectrum of a cosine signal has two impulses placed symmetrically at the frequency of the cosine and its negative.



Fourier transform of any periodic signal

- The Fourier series of a periodic signal $x(t)$ with period T_0 is given by:

$$x(t) = \sum_{-\infty}^{\infty} D_n\, e^{jn\omega_0 t},\ \omega_0 = \frac{2\pi}{T_0}$$

- By taking the Fourier transform on both sides we get:

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n\, \delta(\omega - n\omega_0)$$

Fourier transform of a unit impulse train

- Consider an **impulse train**

$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} \delta\,(t - nT_0)$$

- The Fourier series of this impulse train **can be shown** to be:

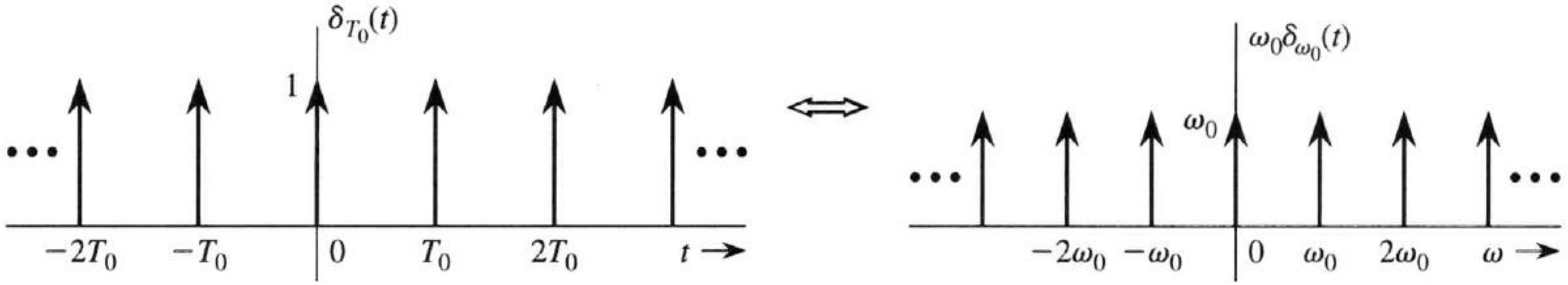
$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} D_n\, e^{jn\omega_0 t} \text{ where } \omega_0 = \frac{2\pi}{T_0} \text{ and } D_n = \frac{1}{T_0}$$

- Therefore, using results from slide 8 we get:

$$X(\omega) = \mathcal{F}\{\delta_{T_0}(t)\} = \frac{1}{T_0}\sum_{-\infty}^{\infty} \mathcal{F}\{e^{jn\omega_0 t}\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} 2\pi\delta(\omega - n\omega_0),\, \omega_0 = \frac{2\pi}{T_0}$$

$$X(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) = \omega_0\, \delta_{\omega_0}(\omega)$$

- The Fourier transform of an impulse train in time (denoted by $\delta_{T_0}(t)$) is an impulse train in frequency (denoted by $\delta_{\omega_0}(\omega)$) .
- The closer (further) the pulses in time the further (closer) in frequency.



Linearity and conjugate properties

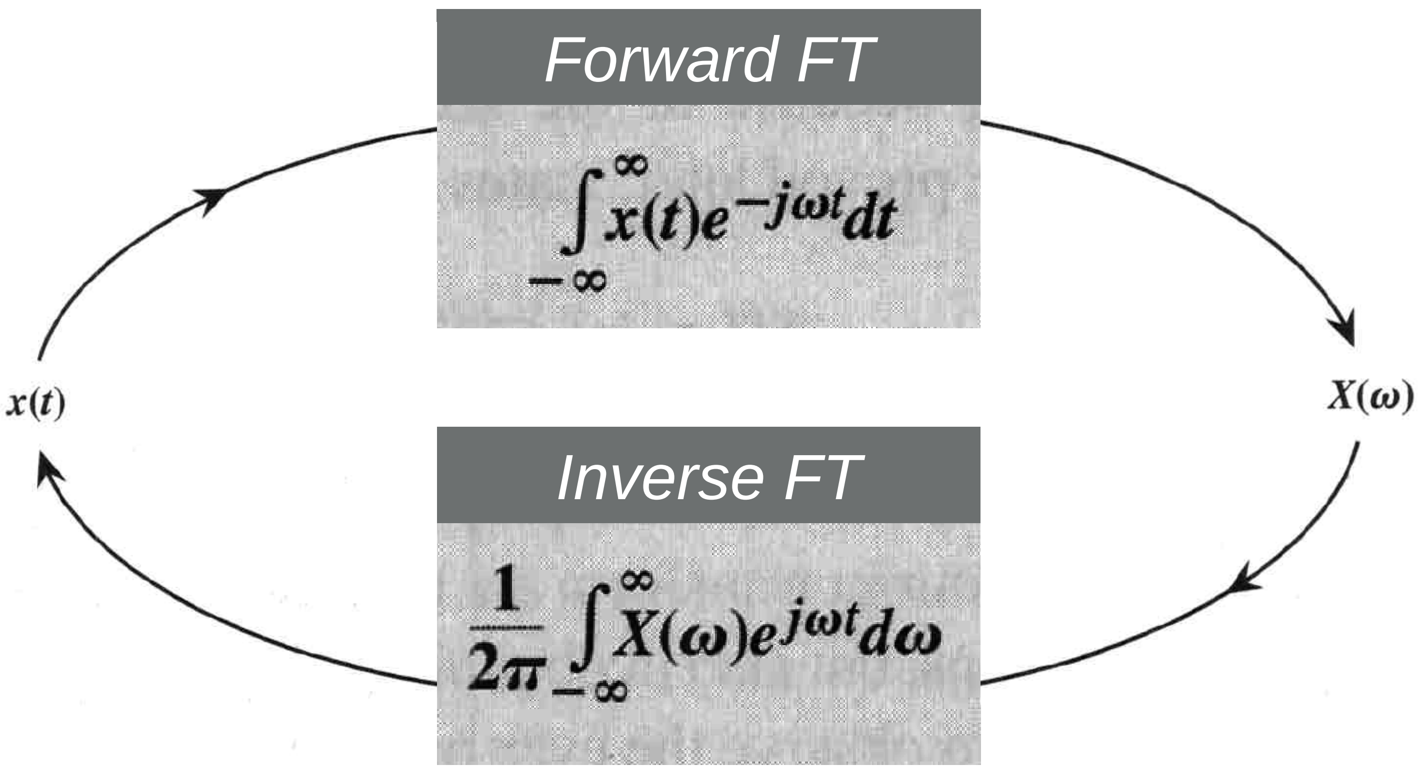
- **Linearity**
  If $x_1(t) \Leftrightarrow X_1(\omega)$ and $x_2(t) \Leftrightarrow X_2(\omega)$, then
  $a_1x_1(t)+a_2x_2(t) \Leftrightarrow a_1X_1(\omega) + a_2X_2(\omega)$

- **Property of conjugate of a signal**
  If $x(t) \Leftrightarrow X(\omega)$ then $x^*(t) \Leftrightarrow X^*(-\omega)$.

- **Property of conjugate symmetry**
  If $x(t)$ is real then $x^*(t) = x(t)$ and therefore, from the property above we see that $X(\omega) = X^*(-\omega)$ or $X(-\omega) = X^*(\omega)$.
  We can write $X(\omega) = A(\omega)e^{j\phi(\omega)}$.
  - $A(\omega), \phi(\omega)$ are the amplitude and phase spectrum respectively. They are real functions.
  - $X^*(\omega) = A(\omega)e^{-j\phi(\omega)}$ and $X^*(-\omega) = A(-\omega)e^{-j\phi(-\omega)}$
  - Based on the last bullet point, for a real function we have:
    $X(\omega) = X^*(-\omega) \Rightarrow A(\omega)e^{j\phi(\omega)} = A(-\omega)e^{-j\phi(-\omega)} \Rightarrow$
    - $A(\omega) = A(-\omega) \Rightarrow$ **for a real signal, the amplitude spectrum is even.**
    - $\phi(\omega) = -\phi(-\omega) \Rightarrow$ **for a real signal, the phase spectrum is odd.**

Time-frequency duality of Fourier transform

- There is a near symmetry between the forward and inverse Fourier transforms.
- The same observation was valid for Laplace transform.



Duality property

- If $x(t) \Leftrightarrow X(\omega)$ then $X(t) \Leftrightarrow 2\pi x(-\omega)$

Proof

From the definition of the inverse Fourier transform we get:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Therefore,

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega$$

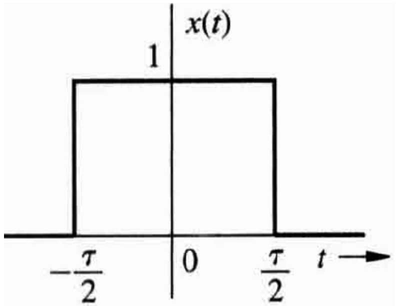
Swapping t with ω and using the definition of forward Fourier transform we have:

$$X(t) \Leftrightarrow 2\pi x(-\omega)$$

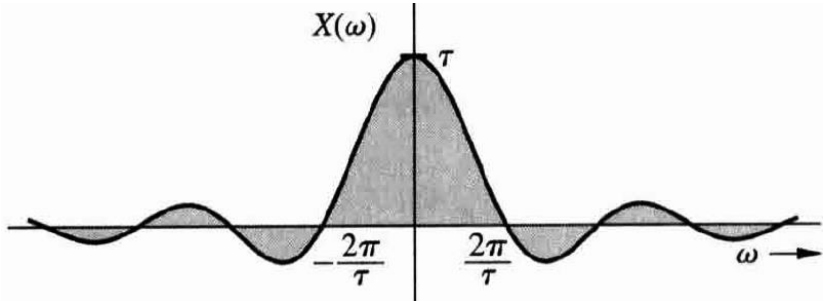
Duality property example

- Consider the Fourier transform of a rectangular function

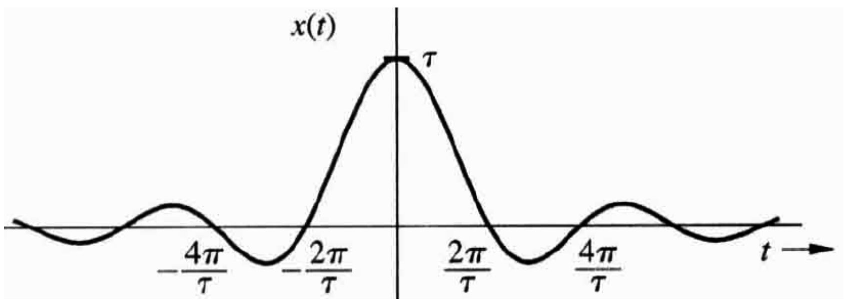
$$\mathrm{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau \mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$



$\Leftrightarrow$



$$\tau \mathrm{sinc}\left(\frac{\tau t}{2}\right) \Leftrightarrow 2\pi\ \mathrm{rect}\left(\frac{-\omega}{\tau}\right) = 2\pi\ \mathrm{rect}\left(\frac{\omega}{\tau}\right)$$



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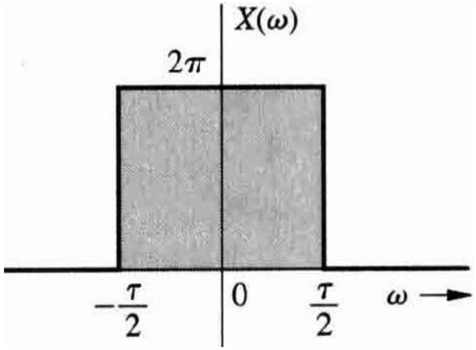
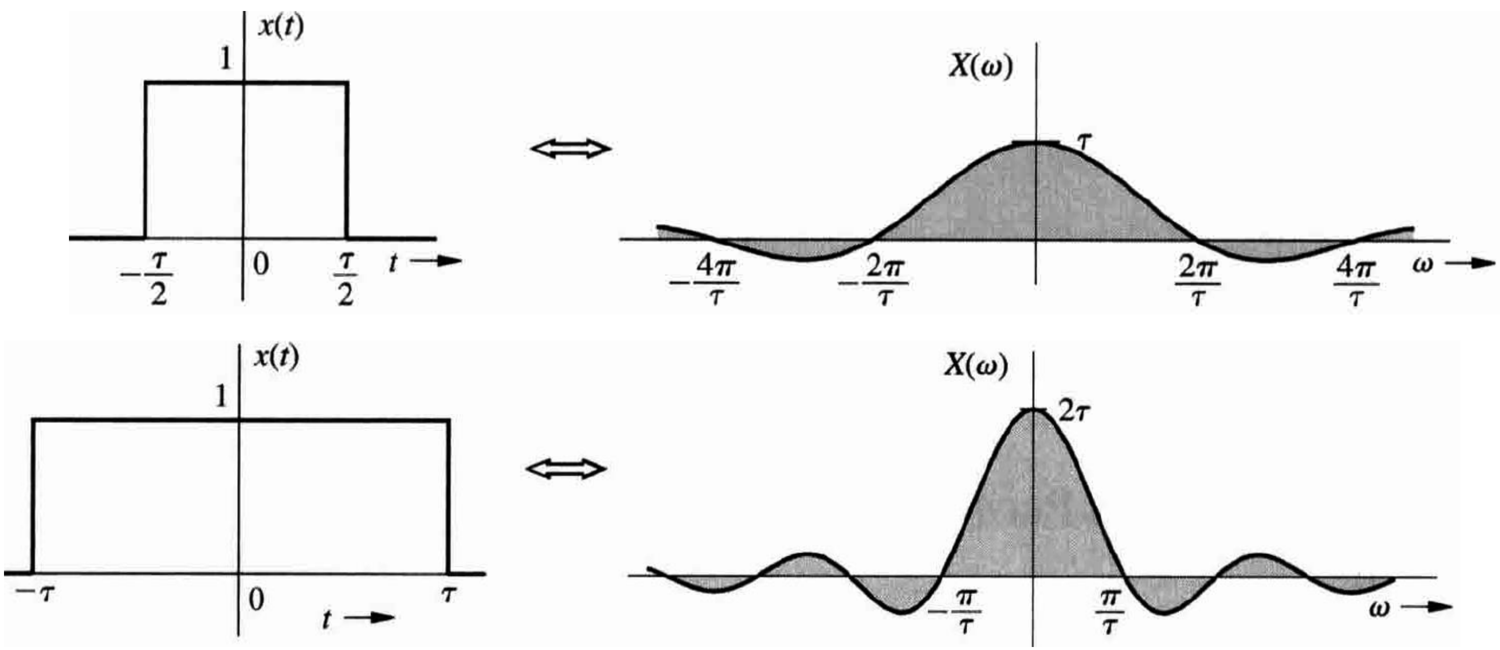


Scaling property

- If $x(t) \Leftrightarrow X(\omega)$ then for any real constant a the following property holds.

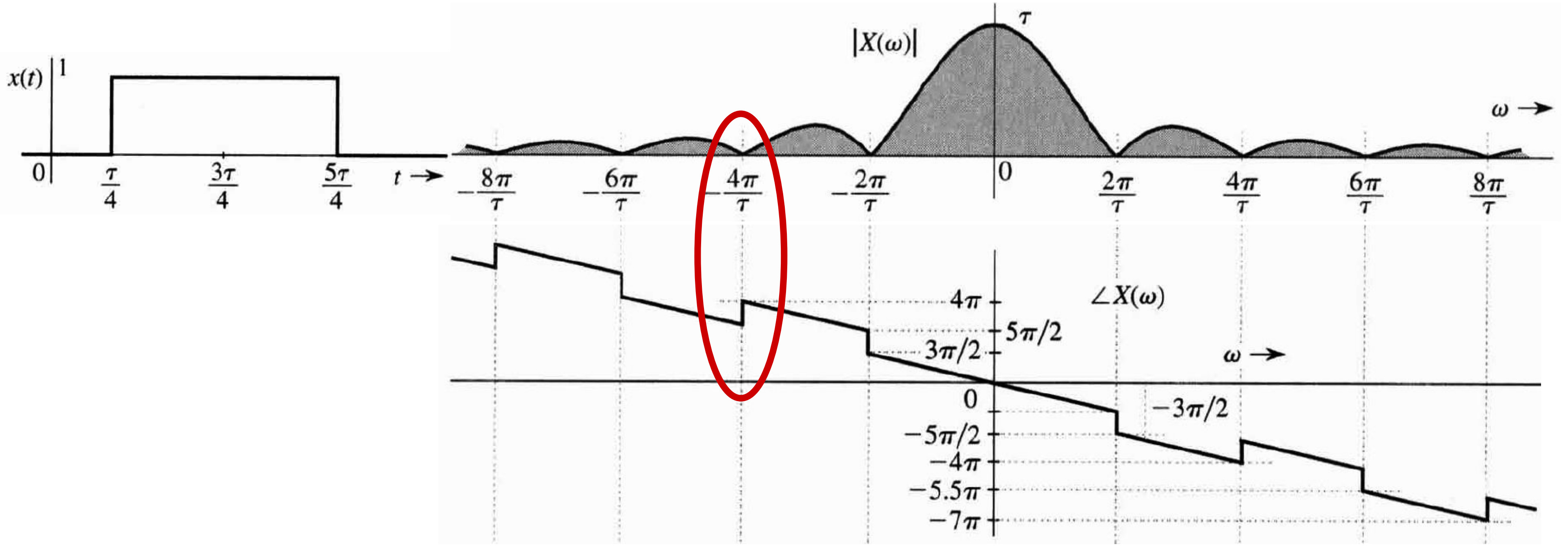
$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

- That is, compression of a signal in time results in spectral expansion and vice versa. As mentioned, the extreme case is the Dirac function and an everlasting constant function.



Time-shifting property with example

- If $x(t) \Leftrightarrow X(\omega)$ then the following property holds.
$$x(t-t_0) \Leftrightarrow X(\omega)\, e^{-j\omega t_0}$$
- Find the Fourier transform of the gate pulse $x(t)$ given by $\mathrm{rect}\left(\frac{t-\frac{3\tau}{4}}{\tau}\right)$.
- By using the time-shifting property we get $X(\omega) = \tau \mathrm{sinc}(\frac{\omega\tau}{2}) e^{-j\omega\frac{3\tau}{4}}$.
- Observe the amplitude (even) and phase (odd) of the Fourier transform.

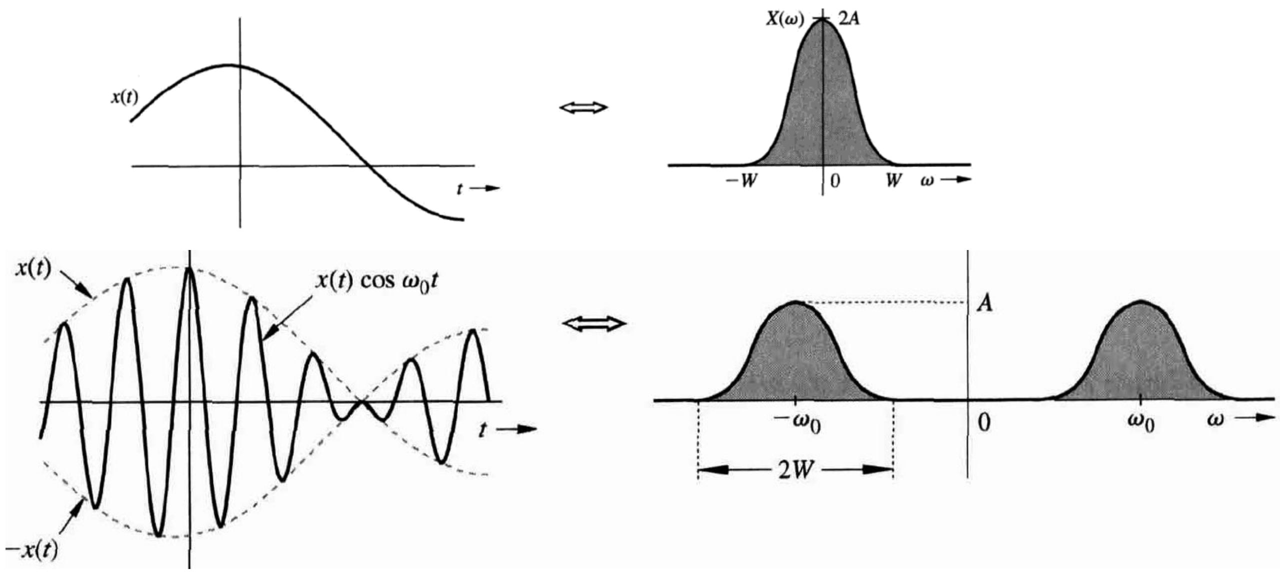


Frequency-shifting property

- If $x(t) \Leftrightarrow X(\omega)$ then $x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega - \omega_0)$. This property states that multiplying a signal by $e^{j\omega_0 t}$ shifts the spectrum of the signal by ω_0.
- In practice, frequency shifting (or amplitude modulation) is achieved by multiplying $x(t)$ by a sinusoid. This is because:

$$x(t)\cos(\omega_0 t) = \frac{1}{2}\left[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right]$$

$$x(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$$



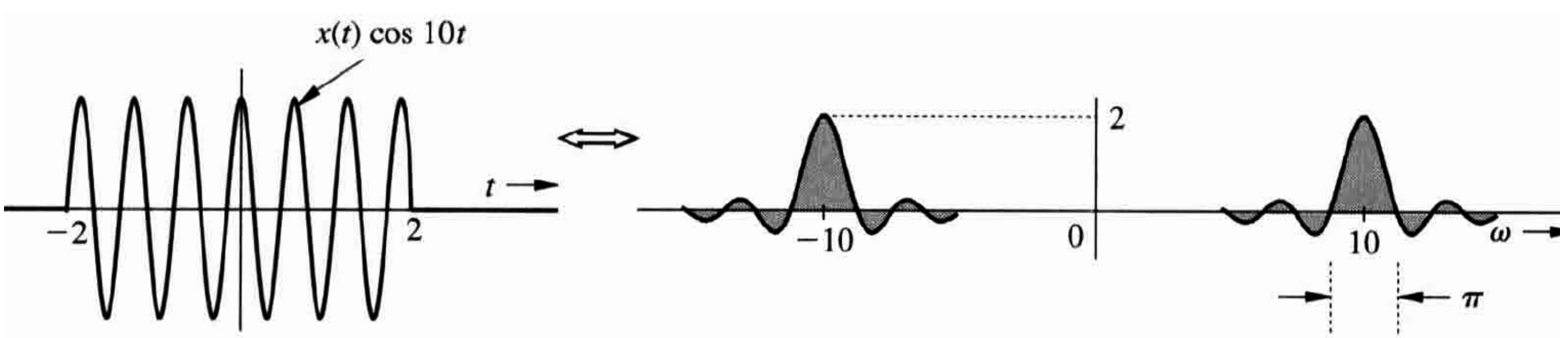
Frequency-shifting example

- Find and sketch the Fourier transform of the signal $x(t)\cos 10t$ where $x(t) = \text{rect}\left(\frac{t}{4}\right)$. We know that $\text{rect}\left(\frac{t}{4}\right) \Leftrightarrow 4\text{sinc}(2\omega)$

$$x(t)\cos(10t) = \frac{1}{2}\left[x(t)e^{j10t} + x(t)e^{-j10t}\right]$$

$$x(t)\cos(10t) \Leftrightarrow \frac{1}{2}\left[X(\omega - 10) + X(\omega + 10)\right]$$

$$x(t)\cos(10t) \Leftrightarrow 2\left\{\text{sinc}[2(\omega - 10)] + \text{sinc}[2(\omega + 10)]\right\}$$



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Is the phase important?

(a)

(b)

Phase from (b), Amp. from (a)

Phase from (a), Amp. from (b)

Convolution properties

- Time and frequency convolution.

  If $x_1(t) \Leftrightarrow X_1(\omega)$ and $x_2(t) \Leftrightarrow X_2(\omega)$, then

  - $x_1(t) * x_2(t) \Leftrightarrow X_1(\omega)X_2(\omega)$
  - $x_1(t)x_2(t) \Leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2\,(\omega)$

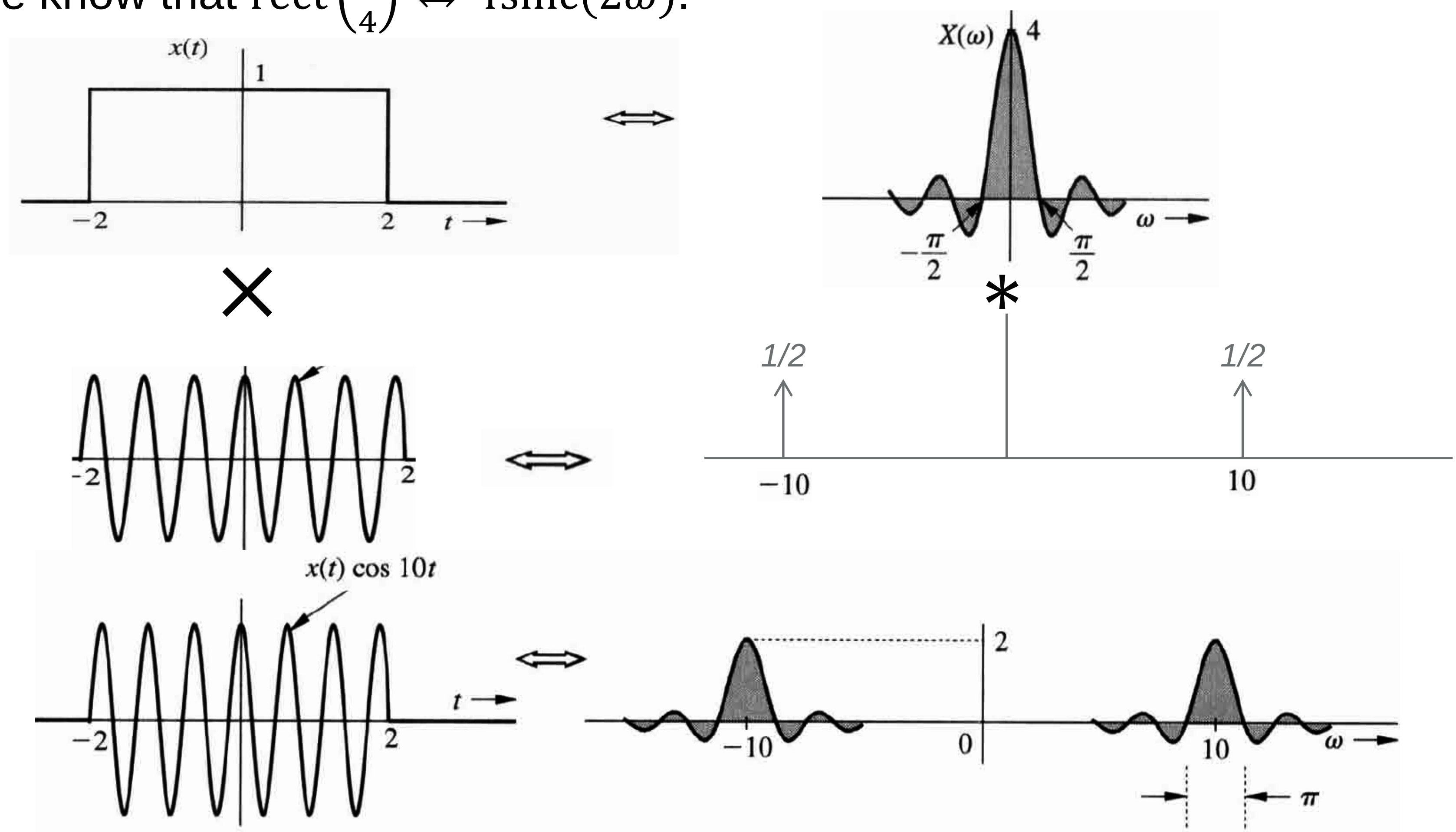
- Let $H(\omega)$ be the Fourier transform of the unit impulse response $h(t)$, i.e.,
$$h(t) \Leftrightarrow H(\omega)$$

- Applying the time-convolution property to $y(t) = x(t) * h(t)$ we get:
$$Y(\omega) = X(\omega)H(\omega)$$

- Therefore, the Fourier Transform of the system's impulse response is the system's Frequency Response.

Frequency convolution example

- Find the spectrum of of the signal $x(t)\cos 10t$ where $x(t) = \mathrm{rect}\left(\frac{t}{4}\right)$.
- We know that $\mathrm{rect}\left(\frac{t}{4}\right) \Leftrightarrow 4\mathrm{sinc}(2\omega)$.



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Time differentiation property

- If $x(t) \Leftrightarrow X(\omega)$ then the following properties hold:
  - Time differentiation property.

    $\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$
  - Time integration property.

    $\int_{-\infty}^{t} x(\tau)d\tau \Leftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

- Compare with the time differentiation property in the Laplace domain.

$$x(t) \Leftrightarrow X(s)$$

$$\frac{dx(t)}{dt} \Leftrightarrow sX(s) - x(0^{-})$$

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Appendix: Proof of the time convolution property

- By definition we have:

$$\mathcal{F}[x_1(t) * x_2(t)] = \int_{t=-\infty}^{\infty} [\int_{\tau=-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau]e^{-j\omega t}dt$$

$$= \int_{\tau=-\infty}^{\infty} [\int_{t=-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-j\omega t}dt]d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)[\int_{t=-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}dt]d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}[\int_{t=-\infty}^{\infty} x_2(t-\tau)e^{-j\omega(t-\tau)}d(t-\tau)]d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}[\int_{t=-\infty}^{\infty} x_2(v)e^{-j\omega v}dv]d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}X_1(\omega)d\tau = X_1(\omega)\int_{\tau=-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau = X_1(\omega)\,X_2(\omega)$$

Fourier transform table 1

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a > 0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	

Fourier transform table 2

No.	$x(t)$	$X(\omega)$	
9	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
11	$u(t)$	$\pi\delta(\omega) + \dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos \omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \dfrac{j\omega}{\omega_0^2 - \omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \dfrac{\omega_0}{\omega_0^2 - \omega^2}$	
15	$e^{-at} \sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
16	$e^{-at} \cos \omega_0 t\, u(t)$	$\dfrac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$

Fourier transform table 3

No.	$x(t)$	$X(\omega)$	
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\mathrm{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\,\mathrm{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\,\mathrm{sinc}\,(Wt)$	$\mathrm{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\,\mathrm{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\,\mathrm{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	

Summary of Fourier transform operations 1

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t) + x_2(t)$	$X_1(\omega) + X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$

Summary of Fourier transform operations 2

Operation	$x(t)$	$X(\omega)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^{t} x(u)\,du$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$