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Signals and Systems

Lecture 10 Final

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Frequency response of a LTI system to an everlasting exponential or a cosine

- We have seen in a previous lecture that a LTI's system response to an everlasting exponential $x(t) = e^{s_0 t}$ is $H(s_0)e^{s_0 t}$.
  [**Proof:** If $h(t)$ is the unit impulse response of an LTI system then:
  $y(t) = h(t) * e^{s_0 t} = \int_{-\infty}^{\infty} h(\tau)e^{s_0(t-\tau)}d\tau = e^{s_0 t} \int_{-\infty}^{\infty} h(\tau)e^{-s_0\tau}d\tau = e^{st}H(s_0)$]
- We represent such **input-output pair** as:

$$e^{s_0 t} \Rightarrow H(s_0)e^{s_0 t}$$

- We set $s_0 = j\omega_0$. This yields the so called **frequency response** evaluated at $\omega = \omega_0$.

$$e^{j\omega_0 t} \Rightarrow H(j\omega_0)e^{j\omega_0 t}$$

$$\cos(\omega_0 t) = \mathrm{Re}\{e^{j\omega_0 t}\} \Rightarrow \mathrm{Re}\{H(j\omega_0)e^{j\omega_0 t}\}$$

- It is often better to express $H(j\omega)$ in polar form as:

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

- Therefore, $H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} = |H(j\omega)|e^{j(\omega t+\angle H(j\omega))}$

$$\cos(\omega_0 t) \Rightarrow |H(j\omega_0)|\cos[\,\omega_0 t + \angle H(j\omega_0)]$$

  - $H(j\omega)$: Frequency response
  - $|H(j\omega)|$: Amplitude response
  - $\angle H(j\omega)$: Phase response

Frequency response of a LTI system to an everlasting exponential or a cosine cont.

- We can also show that a LTI's system response to an everlasting exponential $x(t) = e^{j\omega_0 t+\theta_0}$ is $e^{j\omega_0 t+\theta_0}H(j\omega)$.

Proof:

If $h(t)$ is the unit impulse response of a LTI system then:

$$y(t) = h(t) * e^{j\omega_0 t+\theta_0} = \int_{-\infty}^{\infty} h(\tau)e^{j\omega_0(t-\tau)+\theta_0}d\tau$$

$$= e^{j\omega_0 t+\theta_0}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0\tau}d\tau = e^{j\omega_0 t+\theta_0}H(j\omega_0)$$

- Therefore,

$$\cos(\omega_0 t + \theta_0) \Rightarrow |H(j\omega_0)|\cos[\,\omega_0 t + \theta_0 + \angle H(j\omega_0)]$$

  - $H(j\omega)$: Frequency response
  - $|H(j\omega)|$: Amplitude response
  - $\angle H(j\omega)$: Phase response

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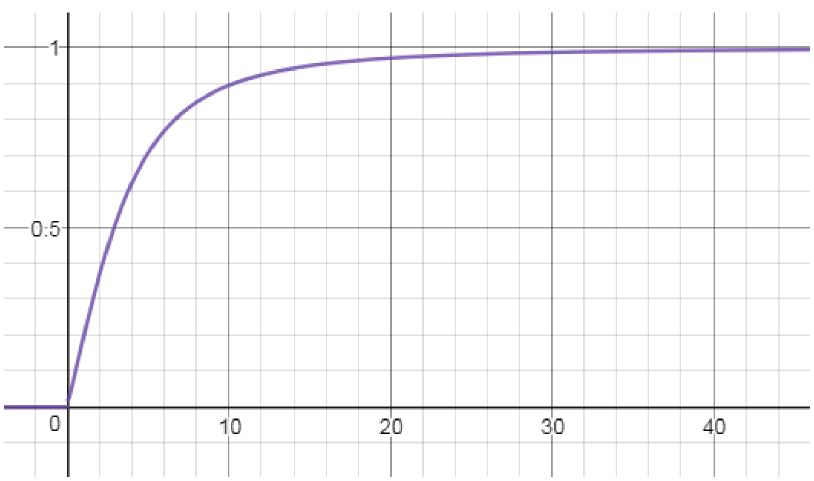
Example

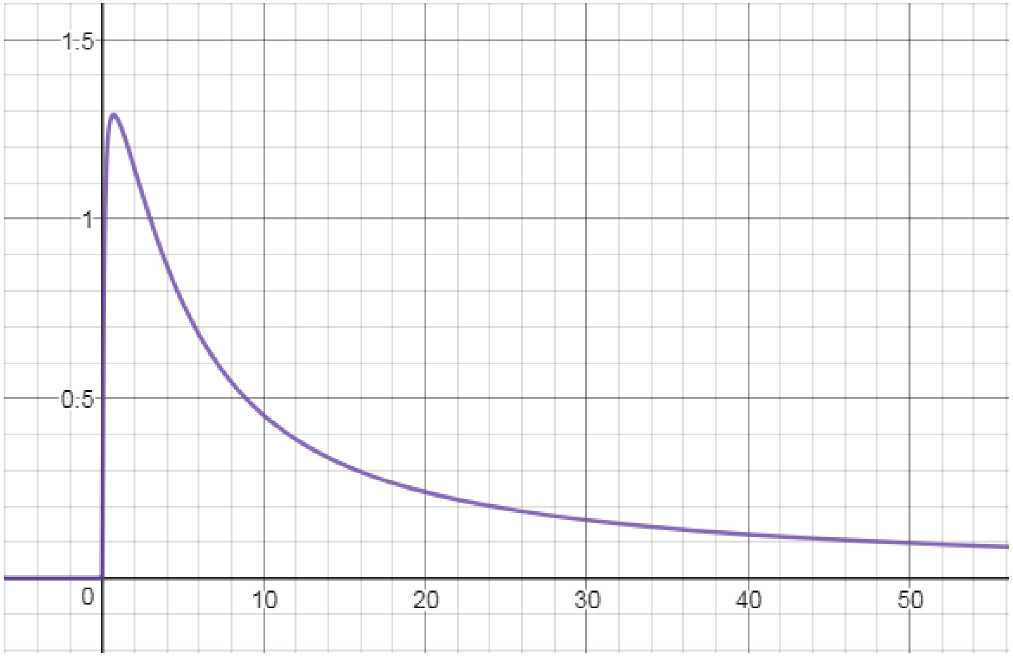
- Find the frequency response (amplitude and phase response) of a system with transfer function:
$$H(s) = \frac{s+0.1}{s+5}$$
Then find the system response $y(t)$ for inputs $x(t) = \cos 2t$ and $x(t) = \cos(10t - 50^{\mathrm{o}})$.

- We substitute $s = j\omega$. Then, we obtain $H(j\omega) = \frac{j\omega+0.1}{j\omega+5}$.
  - Amplitude response: $|H(j\omega)| = \frac{\sqrt{\omega^2+0.01}}{\sqrt{\omega^2+25}}$.
  - Phase response: $\angle H(j\omega) = \Phi(\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$.

Example cont.

- Amplitude response: $|H(j\omega)| = \frac{\sqrt{\omega^2+0.01}}{\sqrt{\omega^2+25}}$.
- Phase response: $\angle H(j\omega) = \Phi(\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$.





Example cont.

- Consider the input $x(t) = \cos 2t$. We have:

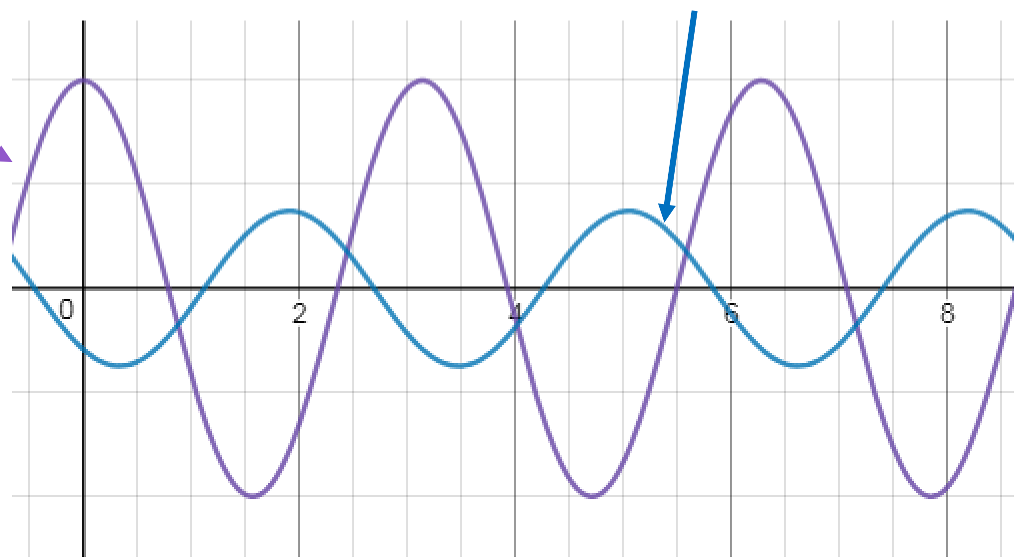
$$\cos(\omega t + \theta) \Rightarrow |H(j\omega)|\cos[\,\omega t + \theta + \angle H(j\omega)]$$

In that case we have $\omega_0 = 2$ and $\theta_0 = 0$.

  - Amplitude response: $|H(j2)| = \frac{\sqrt{2^2+0.01}}{\sqrt{2^2+25}} = 0.372$.
  - Phase response: $\angle H(j2) = \Phi(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 65.3^{\mathrm{o}}$.
- Therefore,

$$y(t) = 0.372\cos(2t + 65.3^{\mathrm{o}})$$

$x(t) = \cos 2t$



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Example cont.

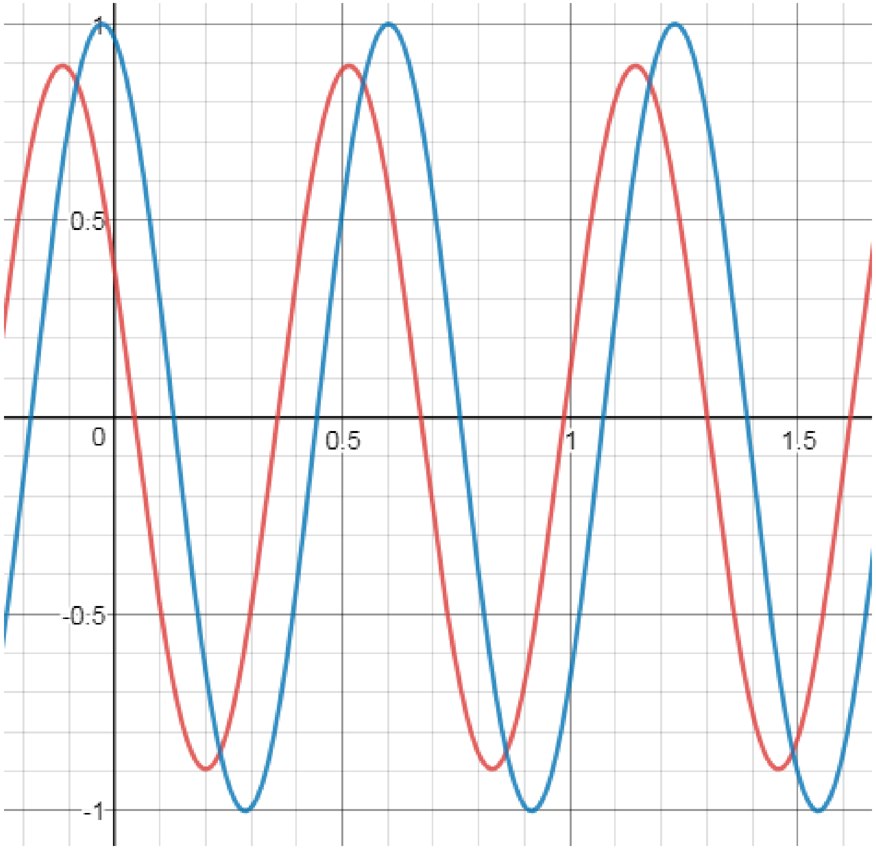
- Consider the input $x(t) = \cos(10t - 50^o)$. We have:

$$\cos(\omega t + \theta) \Rightarrow |H(j\omega)|\cos[\,\omega t + \theta + \angle H(j\omega)]$$

In that case we have $\omega_0 = 10$ and $\theta_0 = -50$.

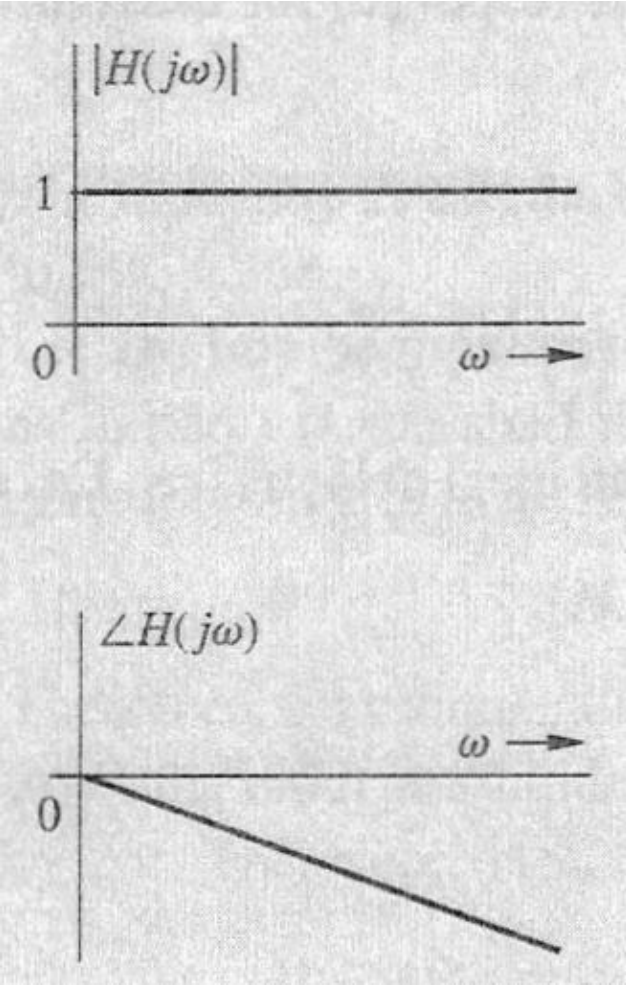
  - Amplitude response: $|H(j10)| = 0.894.$
  - Phase response: $\angle H(j10) = \Phi(j10) = 26^o.$

- Therefore, $y(t) = 0.894\cos(10t - 50^o + 26^o)$



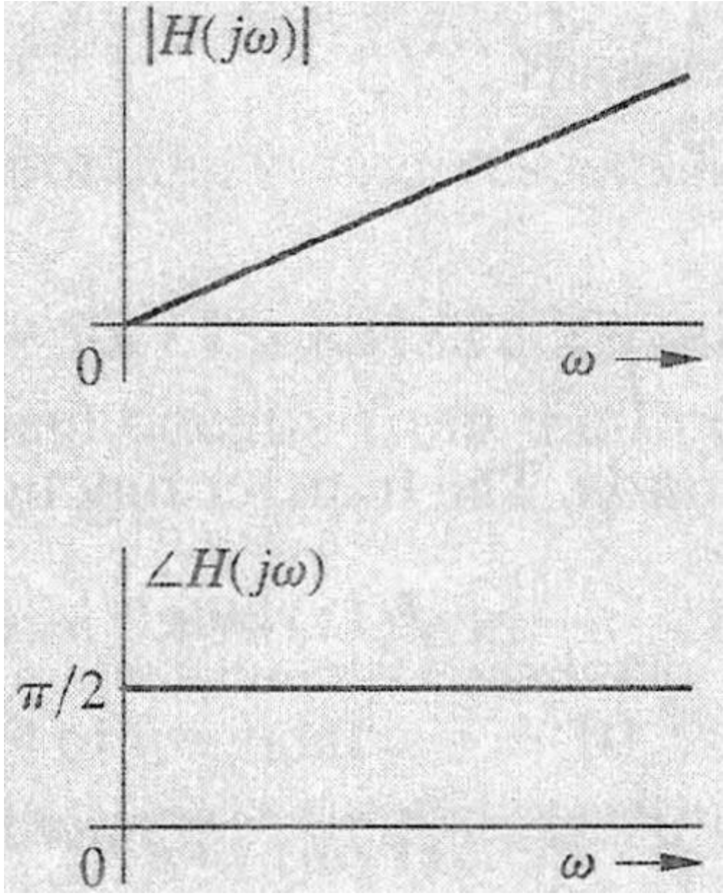
Frequency response of delay of T sec

- The transfer function of an ideal delay is $H(s) = e^{-sT}$.
- Therefore,
  - Amplitude response: $|H(j\omega)| = \left|e^{-j\omega T}\right| = 1.$
  - Phase response: $\angle H(j\omega) = \Phi(j\omega) = -\omega T.$
- Therefore:
  - Delaying a signal by T has no effect on its amplitude.
  - It results in a linear phase shift (with frequency) with a gradient of $-T$.
  - The quantity $-\frac{d\Phi(\omega)}{d\omega} = \tau_g = T$ is known as **Group Delay**.



Frequency response of an ideal differentiator

- The transfer function of an ideal differentiator is $H(s) = s$.
- For $s = j\omega$ we have $H(j\omega) = j\omega = \omega e^{j\frac{\pi}{2}}$.
- Therefore, for $\omega > 0$ we obtain:
  - Amplitude response: $|H(j\omega)| = \omega$.
  - Phase response: $\angle H(j\omega) = \frac{\pi}{2}$.



Example

- Consider the input of an ideal differentiator to be $x(t) = \cos\omega t$.
- The output is:

$$\frac{d}{dt}(\cos\omega t) = -\omega\sin\omega t = \omega\cos(\omega t + \frac{\pi}{2})$$

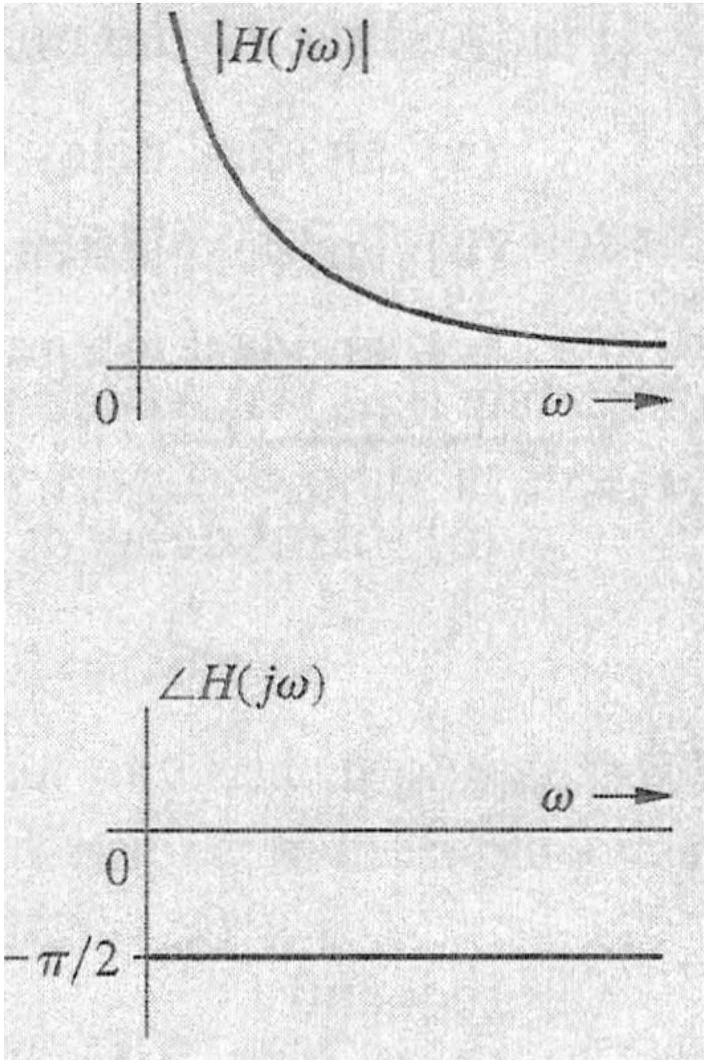
- That's why differentiator is not a nice component to work with; it amplifies high frequency components (i.e., noise).

Frequency response of an ideal integrator

- The transfer function of an ideal integrator is $H(s) = \frac{1}{s}$.
- For $s = j\omega$, $H(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}e^{-j\frac{\pi}{2}}$.
- Therefore, for $\omega > 0$ we obtain:
  - Amplitude response: $|H(j\omega)| = \frac{1}{\omega}$.
  - Phase response: $\angle H(j\omega) = -\frac{\pi}{2}$.
- Consider the input of an ideal integrator to be $x(t) = \cos\omega t$.
- The output is:

$$\int cos\omega t\, dt = \frac{1}{\omega} sin\omega t = \frac{1}{\omega}\cos(\omega t - \frac{\pi}{2})$$

- That's why an integrator is a nice component to work with; it supresses high frequency components (i.e., noise).



Bode Plots: Asymptotic behaviour of amplitude and phase response

- Consider a system with transfer function:

$$H(s) = \frac{K(s+a_1)(s+a_2)}{s(s+b_1)(s^2+b_2s+b_3)} = \frac{Ka_1a_2}{b_1b_3}\frac{(\frac{s}{a_1}+1)(\frac{s}{a_2}+1)}{s(\frac{s}{b_1}+1)(\frac{s^2}{b_3}+\frac{b_2}{b_3}s+1)}$$

- The **poles** are the roots of the denominator polynomial. In this case, the poles of the system are $s = 0$, $s = -b_1$ and the solutions of the quadratic

$$s^2 + b_2s + b_3 = 0$$

Which we assume to form a complex conjugate pair.

- The **zeros** are the roots of the numerator polynomial. In this case, the zeros of the system are: $s = -a_1$, $s = -a_2$.

Bode Plots: Asymptotic behaviour of amplitude and phase response

- Now let $s = j\omega$. The amplitude response $|H(j\omega)|$ can be rearranged as:

$$|H(j\omega)| = \frac{Ka_1a_2}{b_1b_3}\frac{\left|1+\frac{j\omega}{a_1}\right|\left|1+\frac{j\omega}{a_2}\right|}{|j\omega|\left|1+\frac{j\omega}{b_1}\right|\left|1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right|}$$

- We express the above in decibel (i.e., $20\log(\cdot)$):

$$20\log|H(j\omega)| = 20\log\frac{Ka_1a_2}{b_1b_3} + 20\log\left|1+\frac{j\omega}{a_1}\right| + 20\log\left|1+\frac{j\omega}{a_2}\right|$$
$$-20\log|j\omega| - 20\log\left|1+\frac{j\omega}{b_1}\right| - 20\log\left|1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right|$$

- By imposing a log operation the amplitude response (in dB) is broken into building block components that are added together.
- We have three types of building block terms: A term $j\omega$, a first order term $1+\frac{j\omega}{a}$ and a second order term with complex conjugate roots.

Advantages of logarithmic units

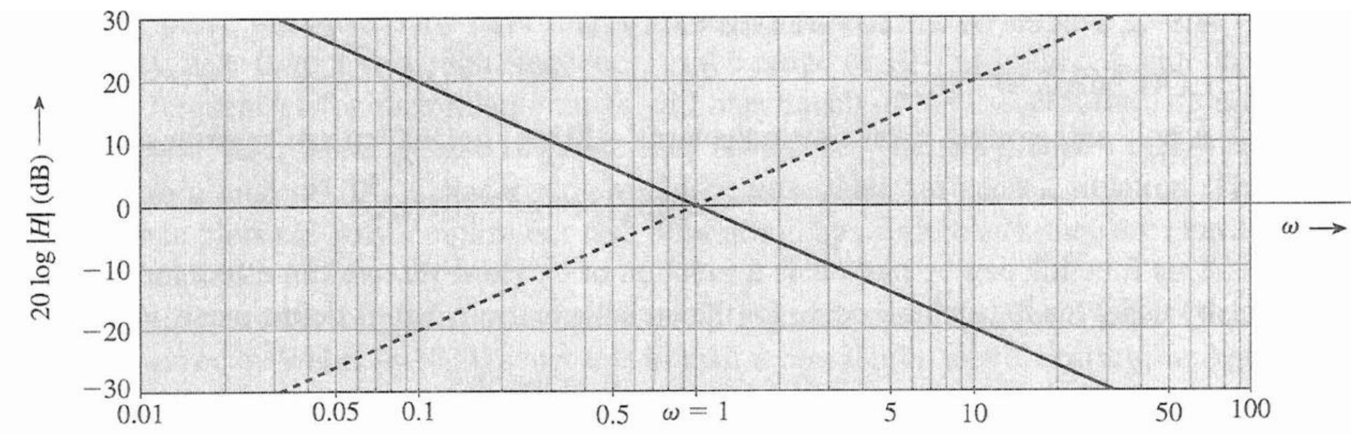
- They are desirable in several applications, where the variables considered have a very large range of values.
- The above is particularly true in frequency response amplitude plots since we require to plot frequency response from 10^{-6} to 10^{6} or higher.
- A plot of such a large range on a linear scale will bury much of the useful information at lower frequencies.
- In humans the relationship between stimulus and perception is logarithmic.
  - This means that if a stimulus varies as a geometric progression (i.e., multiplied by a fixed factor), the corresponding perception is altered in an arithmetic progression (i.e., in additive constant amounts). For example, if a stimulus is tripled in strength (i.e., 3 x 1), the corresponding perception may be two times as strong as its original value (i.e., 1 + 1).
  - There is behind the above observations developed by Weber and Frechner.

Building blocks for Bode plots
a pole at the origin: amplitude response

- A pole at the origin $1/s$ contributes to the frequency response with the term $-20\log|j\omega| = -20\log\omega$.
  - We can effect further simplification by using the logarithmic function for the variable ω itself. Therefore, we define $u = \log\omega$.
- Therefore, $-20\log\omega = -20u$.
  - The above is a straight line with a slope of -20.
  - A ratio of 10 in ω is called a **decade**. If $\omega_2 = 10\omega_1$ then
    $u_2 = \log\omega_2 = \log 10\omega_1 = \log 10 + \log\omega_1 = 1 + \log\omega_1 = 1 + u_1$
    $-20\log\omega_2 = -20\log 10\omega_1 = -20\log 10 - 20\log\omega_1 = -20 - 20\log\omega_1$
  - A ratio of 2 in ω is called an **octave**. If $\omega_2 = 2\omega_1$ then
    $u_2 = \log\omega_2 = \log 2\omega_1 = \log 2 + \log\omega_1 = 0.301 + \log\omega_1 = 0.301 + u_1$
    $-20\log\omega_2 = -20\log 2\omega_1 = -20\log 2 - 20\log\omega_1 = -6.02 - 20\log\omega_1$
- **Based on the above, equal increments in u are equivalent to equal ratios in ω.**
- The amplitude response plot has a slope of $-20\mathrm{dB/decade}$ or $-20(0.301) = -6.02\mathrm{dB/octave}$.
- The amplitude plot crosses the ω axis at $\omega = 1$, since $u = \log\omega = 0$ for $\omega = 1$.

Bode plots – a zero at the origin: amplitude

- A zero at the origin s contributes to the frequency response with the term $20\log|j\omega| = 20\log\omega = 20u$.
- The amplitude plot has a slope of $20dB/$ decade or $20(0.301) = 6.02dB/$octave.
- The amplitude plot for a zero at the origin is a mirror image about the ω axis of the plot for a pole at the origin.
- As you can see the horizontal axis depicts $u = \log\omega$ and not ω.



Amplitude responses of a pole (solid line) and a zero (dotted line) at the origin

Bode plots – first order pole: amplitude response

- The log amplitude of a first order pole at $-a$ is $-20\log\left|1+\frac{j\omega}{a}\right|$.
  - $\omega \ll a \Rightarrow -20\log\left|1+\frac{j\omega}{a}\right| \approx -20\log 1 = 0$
  - $\omega \gg a \Rightarrow -20\log\left|1+\frac{j\omega}{a}\right| \approx -20\log(\frac{\omega}{a}) = -20\log\omega + 20\log a$

    This represents a straight line (when plotted as a function of u, the log of ω) with a slope of $-20dB/$decade or $-20(0.301) = -6.02dB/$octave. When $\omega = a$ the log amplitude is zero. Hence, this line crosses the ω axis at $\omega = a$. Note that the asymptotes meet at $\omega = a$.
- The exact log amplitude for this pole is:

$$-20\log\left|1+\frac{j\omega}{a}\right| = -20\log\left(1+\frac{\omega^2}{a^2}\right)^{\frac{1}{2}} = -10\log\left(1+\frac{\omega^2}{a^2}\right)$$

- The maximum error between the actual and asymptotic plots occurs at $\omega = a$ called **corner frequency** or **break frequency.** This error is:

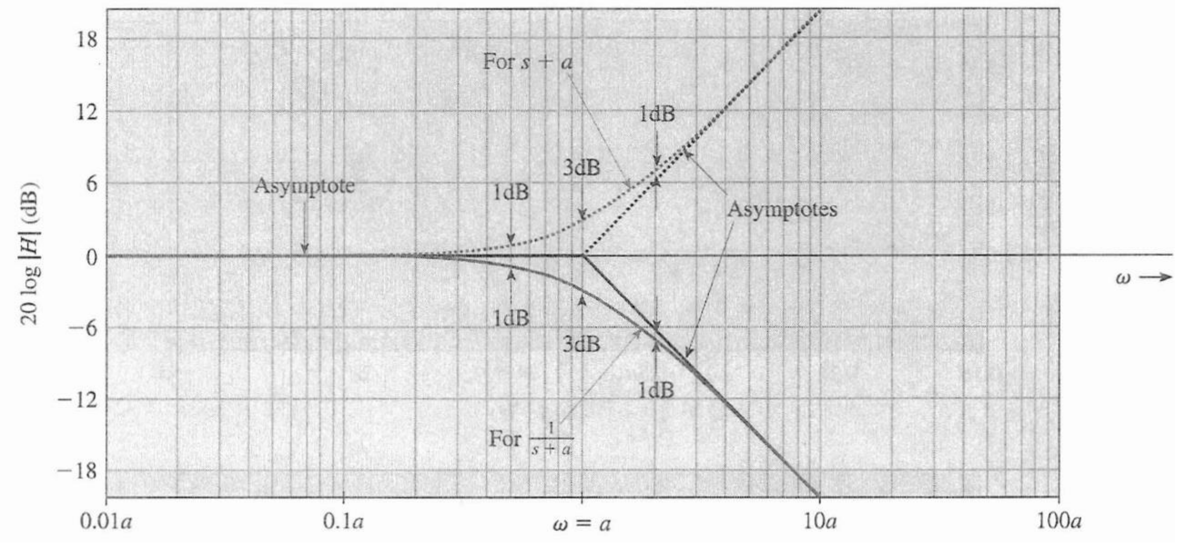
  $\mathbf{err}(\boldsymbol{a}) = \mathbf{exact\_amplitude}(\boldsymbol{a}) - \mathbf{asymptotic\_amplitude}(\boldsymbol{a})$

  $= -\mathbf{10log}\left(\mathbf{1}+\frac{\boldsymbol{a}^2}{\boldsymbol{a}^2}\right) - \mathbf{0} = -\mathbf{10log2} = -\mathbf{3}\boldsymbol{d}\mathbf{B}.$

Bode plots – first order zero: amplitude

- A first order zero at $-a$ gives rise to the term $20\log\left|1+\frac{j\omega}{a}\right|$.
  - $\omega \ll a \Rightarrow 20\log\left|1+\frac{j\omega}{a}\right| \approx 20\log 1 = 0.$
  - $\omega \gg a \Rightarrow 20\log\left|1+\frac{j\omega}{a}\right| \approx 20\log(\frac{\omega}{a}) = 20\log\omega - 20\log a.$

    This represents a straight line with a slope of $20d\mathrm{B}/$decade.
- The amplitude plot for a zero at $-a$ is a mirror image about the ω axis of the plot for a pole at $-a$.



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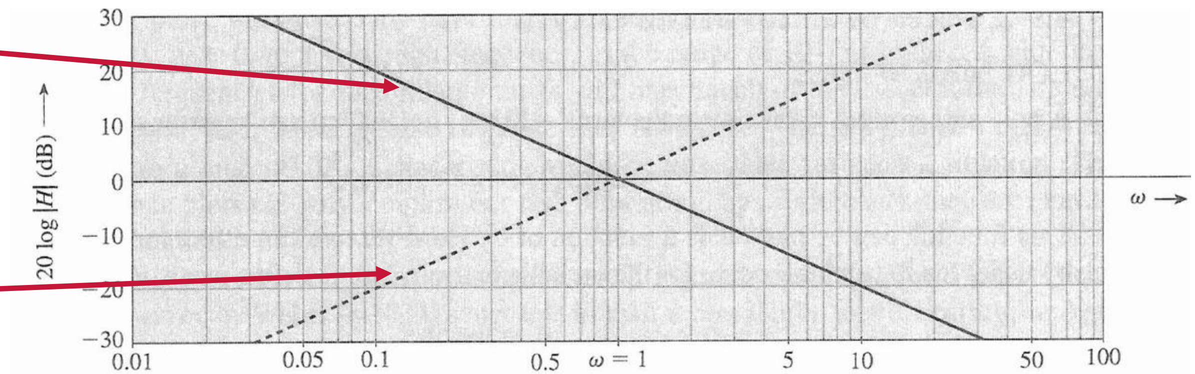
Summary of first order building blocks for Bode plots: amplitude

- Pole term: $-20\log|j\omega|$

  $= -20\log\omega$

- Zero term: $20\log|j\omega|$

  $= 20\log\omega$



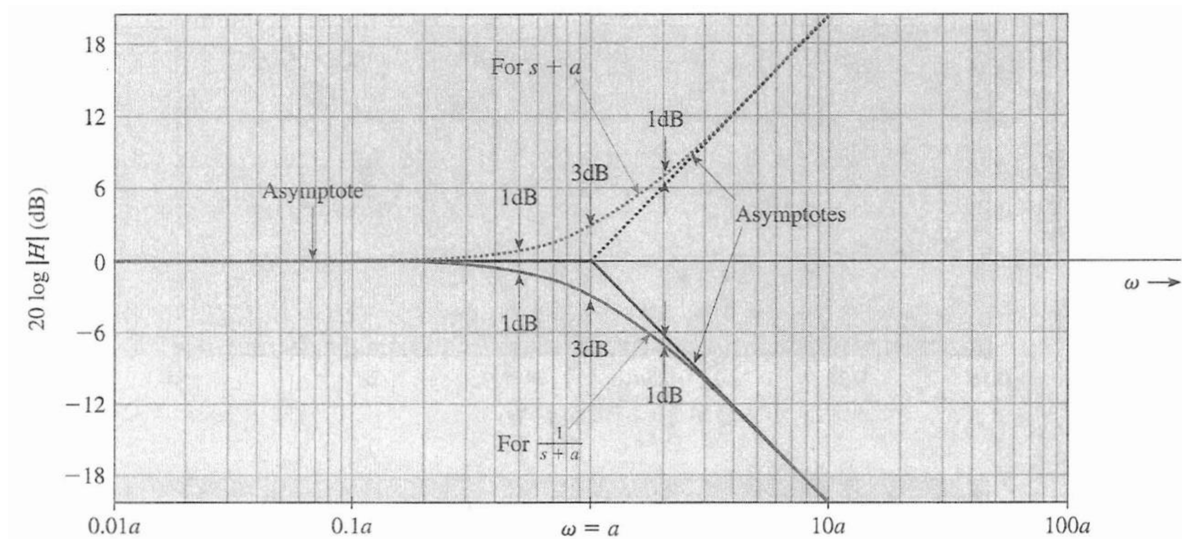
- Pole term: $-20\log\left|1+\frac{j\omega}{a}\right|$
  - $\omega \ll a \Rightarrow -20\log\left|1+\frac{j\omega}{a}\right|$

    $\approx -20\log 1 = 0$
  - $\omega \gg a \Rightarrow -20\log\left|1+\frac{j\omega}{a}\right|$

    $\approx -20\log(\frac{\omega}{a})$

    $= -20\log\omega + 20\log a$
  - $\omega = a$

    $-20\log|1+j| = -20\log\sqrt{2} \approx -3db$



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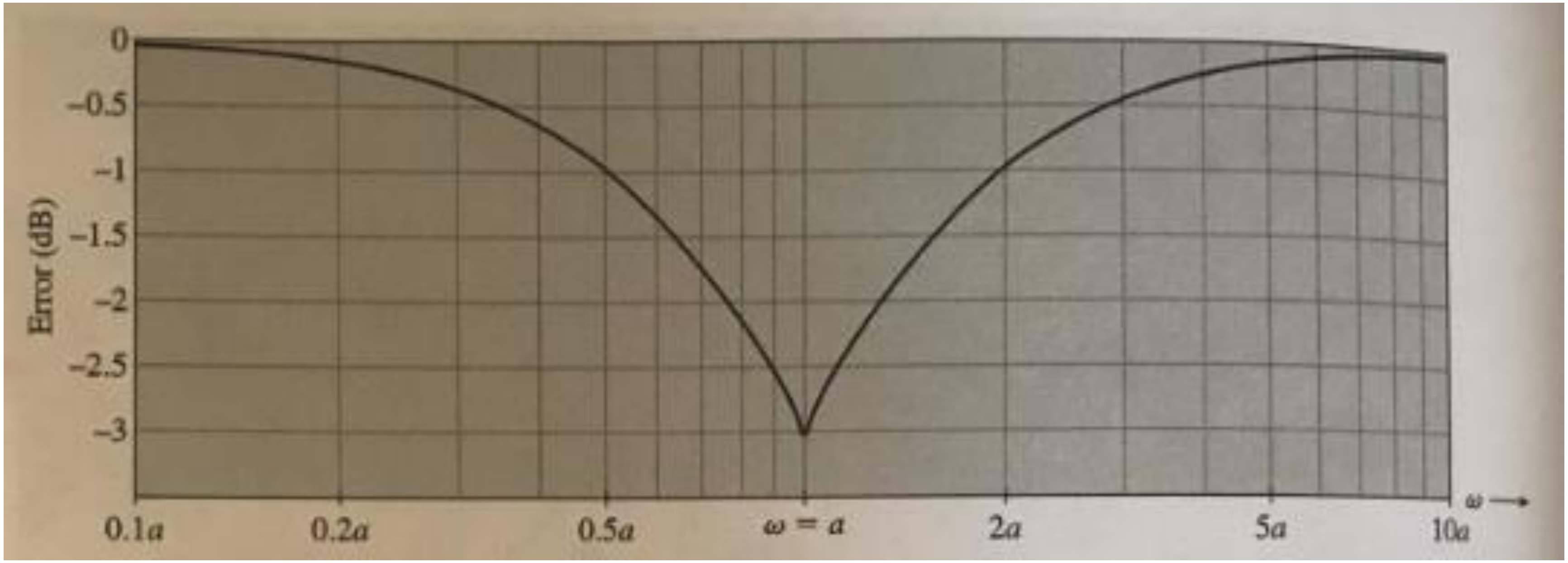
Error in the asymptotic approximation of amplitude due to a first order pole

- The error of the approximation as a function of ω is shown in the figure below.
- The actual plot can be obtained if we add the error to the asymptotic plot.
- **Problem:** Find the error when the frequency is equal to the corner frequency and 2, 5 and 10 times larger or smaller.

  (**Answers**: $-3dB, -1dB, -0.17dB,$ negligible. See subsequent slides.)



Error in the asymptotic approximation of amplitude due to a first order pole

- The error of the approximation of the true response with an asymptotic line at a frequency ω for the case of a first order pole is:

$$\mathrm{err}(\omega) = \mathrm{true}(\omega) - \mathrm{asymptotic}(\omega)$$

$$= -10\log\left(1+\frac{\omega^2}{a^2}\right) - (-20\log\omega + 20\log a)$$

$$= -10\log\left(1+\frac{\omega^2}{a^2}\right) + 20\log\omega - 20\log a$$

- For $\omega = a$ **(also shown at bottom of Slide 16)**

$$\mathrm{err}(\omega) = -10\log\left(1+\frac{a^2}{a^2}\right) + 20\log a - 20\log a = -10\log(2) = -10\cdot 0.3$$

$$= -3\mathrm{dB}$$

- For $\omega = 2a$,

$$\mathrm{err}(\omega) = -10\log\left(1+\frac{4a^2}{a^2}\right) + 20\log 2a - 20\log a =$$

$$= -10\log(5) + 20\log(2) = -10\cdot 0.7 + 20\cdot 0.3 = -1\mathrm{dB}$$

Bode plots – error of the approximation cont.

$$\mathrm{err}(\omega) = -10\log\left(1 + \frac{\omega^2}{a^2}\right) + 20\log\omega - 20\log a$$

- For $\omega = 5a$,

$$\mathrm{err}(\omega) = -10\log\left(1 + \frac{25a^2}{a^2}\right) + 20\log 5a - 20\log a = -10\log(26) + 20\log(5)$$

$$= -10 \cdot 1.41497334 + 20 \cdot 0.69897 = -0.17\mathrm{dB}$$

For $\omega = 10a$,

$$\mathrm{err}(\omega) = -10\log\left(1 + \frac{100a^2}{a^2}\right) + 20\log 10a - 20\log a$$

$$= -10\log(101) + 20\log(10) = -20.0432 + 20 \cdot 1 \cong 0$$

Bode plots – second order pole : amplitude

- Now consider the quadratic term: $s^2 + b_2 s + b_3$.
- It is quite common to express the above term as:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right) = \omega_n^2[1 + 2\zeta(\frac{s}{\omega_n}) + (\frac{s}{\omega_n})^2]$$

  - The scalar ζ is called **damping factor**.
  - The scalar ω_n is called **natural frequency**.
- The log amplitude response is obtained by setting $s = j\omega$ and taking the magnitude.
- For the time being we are not interested in the gain ω_n^2.
- log amplitude $= -20\log\left|1 + 2j\zeta(\frac{\omega}{\omega_n}) + (\frac{j\omega}{\omega_n})^2\right|$
  - $\omega \ll \omega_n$, log amplitude $\approx -20\log 1 = 0$
  - $\omega \gg \omega_n$, log amplitude $\approx -20\log\left|-(\frac{\omega}{\omega_n})^2\right| = -40\log(\frac{\omega}{\omega_n})$

    $= -40\log\omega + 40\log\omega_n = -40u + 40\log\omega_n$

The exact log amplitude is $-20\log\left\{[1 - (\frac{\omega}{\omega_n})^2]^2 + 4\zeta^2(\frac{\omega}{\omega_n})^2\right\}^{1/2}$

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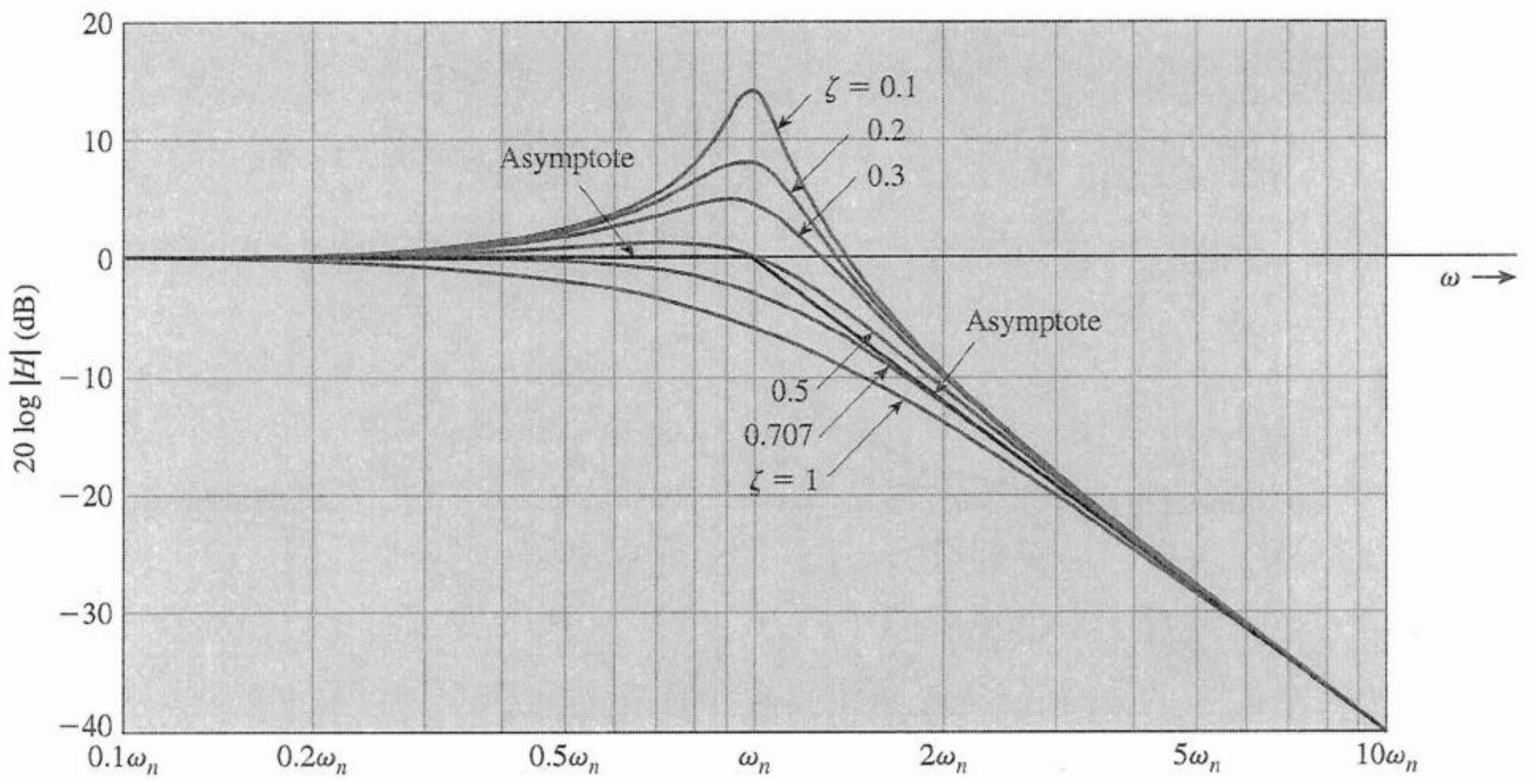
Bode plots – second order pole : amplitude

- The log amplitude involves a parameter ζ called **damping factor**, resulting in a different plot for each value of ζ.
- It can be proven that for complex-conjugate poles $\zeta < 1$.
- For $\zeta \geq 1$, the two poles in the second order factor are not longer complex but real, and each of these two real poles can be dealt with as a separate first order factor.
- The amplitude plot for a pair of complex conjugate zeros is a mirror image about the ω axis of the plot for a pair of complex conjugate poles.

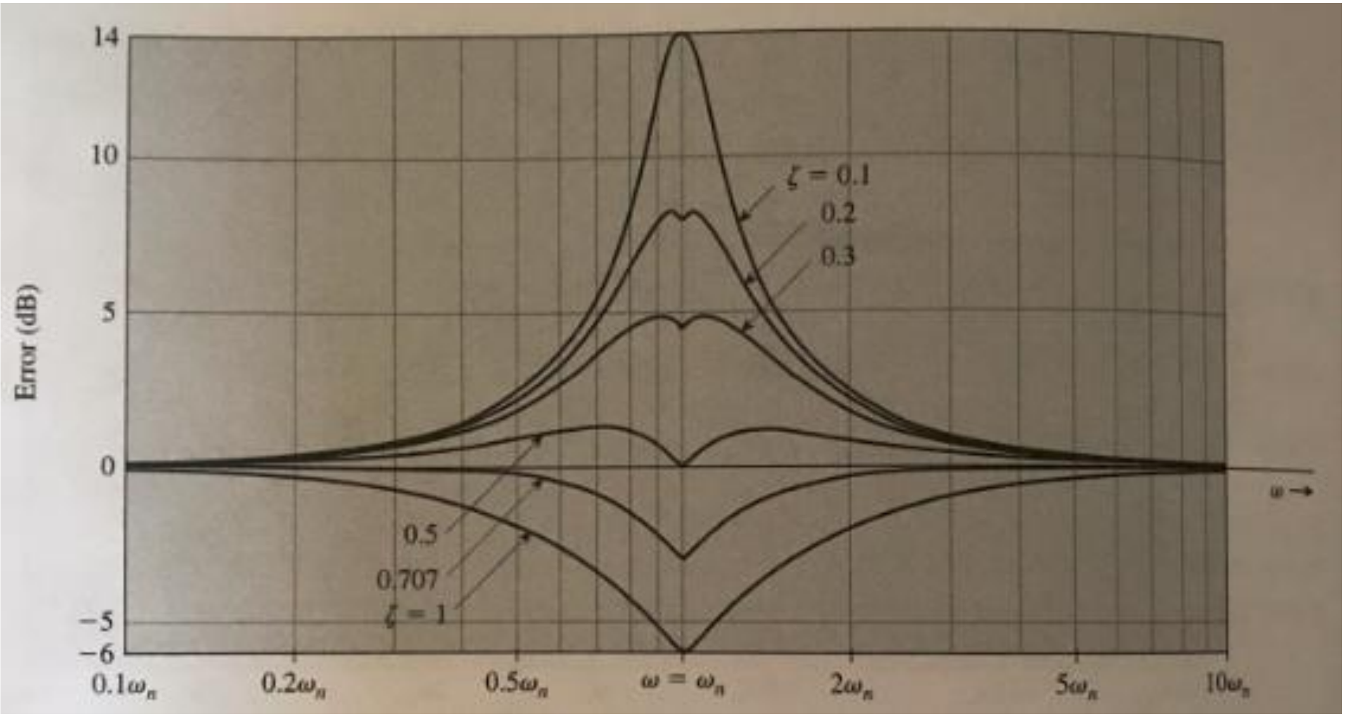
Bode plots – second order pole : amplitude

- The exact log amplitude is $= -20\log\left\{[1-(\frac{\omega}{\omega_n})^2]^2+4\zeta^2(\frac{\omega}{\omega_n})^2\right\}^{1/2}$



Error in the asymptotic approximation of amplitude due to a pair of complex conjugate poles

- The error of the approximation as a function of ω is shown in the figure below for various values of ζs.
- The actual plot can be obtained if we add the error to the asymptotic plot.



Bode plots example: amplitude

- Consider a system with transfer function:

$$H(s) = \frac{20s(s+100)}{(s+2)(s+10)}$$

$$H(s) = \frac{20\times100}{2\times10}\,\frac{s(1+\frac{s}{100})}{(1+\frac{s}{2})(1+\frac{s}{10})} = 100\,\frac{s(1+\frac{s}{100})}{(1+\frac{s}{2})(1+\frac{s}{10})}$$

- **Step 1:** Since the constant term is 100 or $20\log100 = 40\text{dB}$, we relabel the horizontal axis as the 40dB line.
- **Step 2:** For each pole and zero term draw an asymptotic plot.
- **Step 3:** Add all the asymptotes.
- **Step 4:** Apply corrections if possible. For corrections we use the frequencies 1, 2, 10, 100. They are mostly corner frequencies.

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Bode plots example: amplitude. Corrections.

- **Correction at $\boldsymbol{\omega = 1}$**
  - Due to corner frequency at $\omega = 2$ is $-1dB$.
  - Due to corner frequency at $\omega = 10$ is negligible.
  - Due to corner frequency at $\omega = 100$ is negligible.

  Total correction at $\omega = 1$ is $-1dB$.
- **Correction at $\boldsymbol{\omega = 2}$**
  - Due to corner frequency at $\omega = 2$ is $-3dB$.
  - Due to corner frequency at $\omega = 10$ is $-0.17dB$.
  - Due to corner frequency at $\omega = 100$ is negligible.

  Total correction at $\omega = 2$ is $-3.17dB$.
- **Correction at $\boldsymbol{\omega = 10}$**
  - Due to corner frequency at $\omega = 10$ is $-3dB$.
  - Due to corner frequency at $\omega = 2$ is $-0.17dB$.
  - Due to corner frequency at $\omega = 100$ is negligible.

  Total correction at $\omega = 10$ is $-3.17dB$.

Bode plots example: amplitude. Corrections cont.

- **Correction at $\boldsymbol{\omega = 100}$**
  - Due to corner frequency at $\omega = 100$ is $3dB$.
  - Due to corner frequency at $\omega = 2$ is negligible.
  - Due to corner frequency at $\omega = 10$ is negligible.

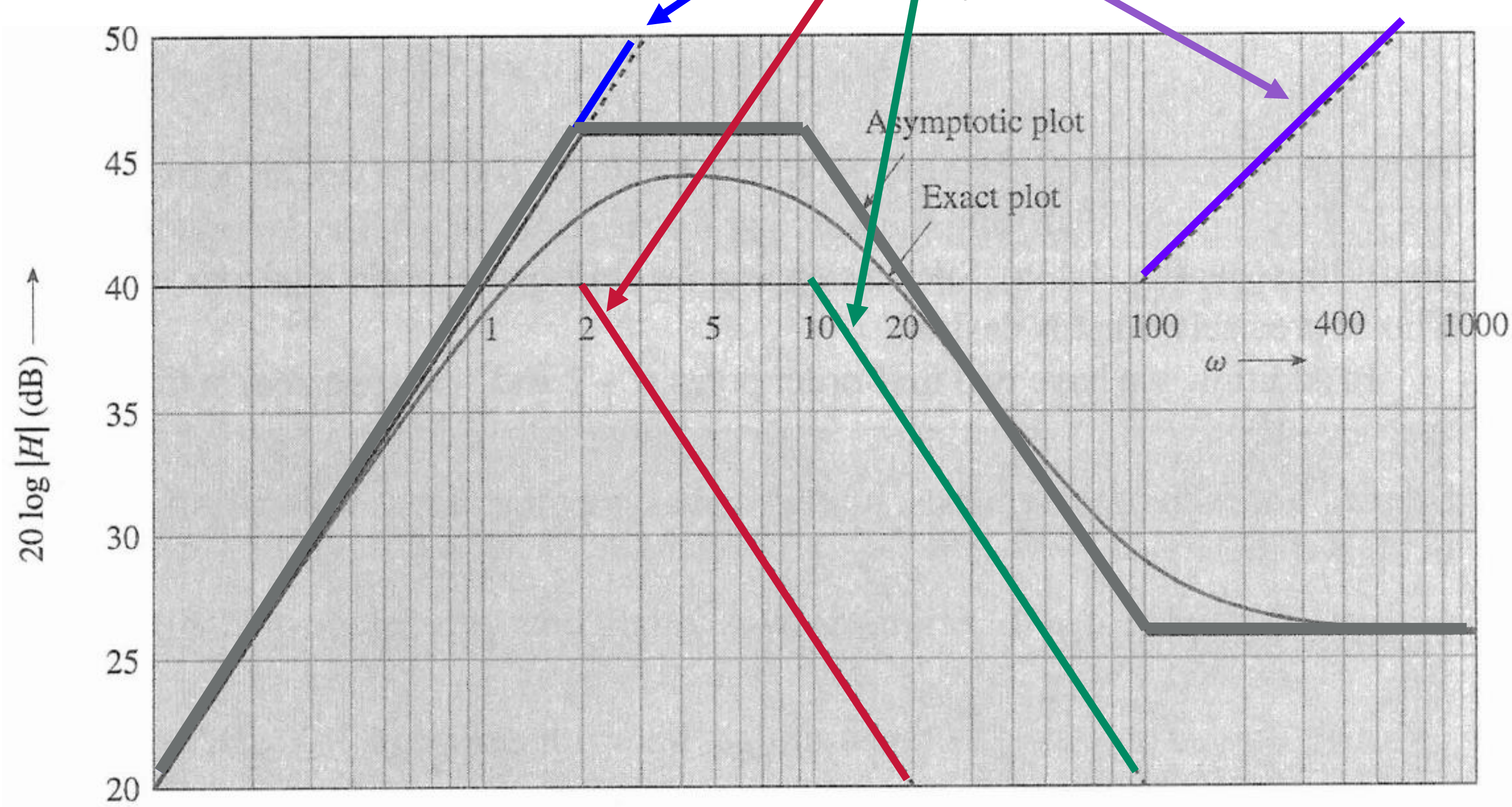
  Total correction at $\omega = 100$ is $3dB$.

- Correction at intermediate points other than corner frequencies may be considered for more accurate plots.

Bode plots example: total amplitude

- Observe now the final plot for the previous system with transfer function:

$$H(s) = 100\frac{s(1+\frac{s}{100})}{(1+\frac{s}{2})(1+\frac{s}{10})}$$



Bode plots: phase

- Now consider the phase response for the earlier transfer function:

$$H(j\omega) = \frac{Ka_1a_2}{b_1b_3}\frac{(1+\frac{j\omega}{a_1})(1+\frac{j\omega}{a_2})}{j\omega(1+\frac{j\omega}{b_1})(1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_3})}$$

- The phase response is:

$$\angle H(j\omega) = \angle\left(1+\frac{j\omega}{a_1}\right) + \angle\left(1+\frac{j\omega}{a_2}\right) - \angle j\omega$$
$$-\angle(1+\frac{j\omega}{b_1}) - \angle(1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_3})$$

- Again, we have three types of terms.

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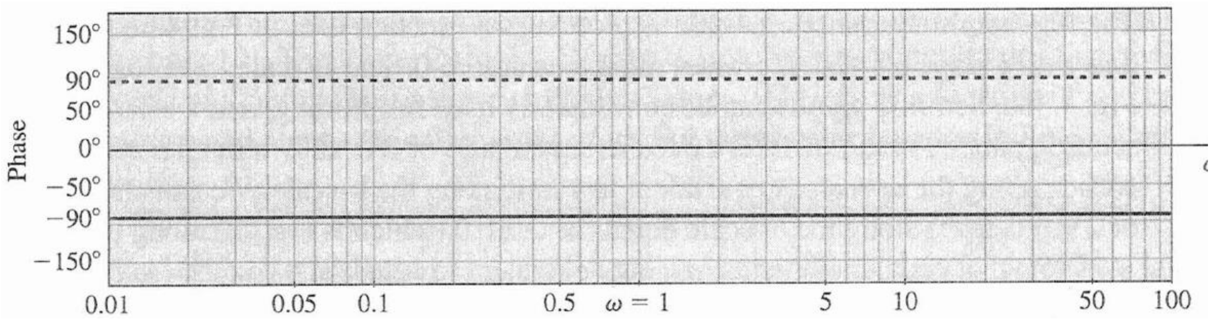
Bode plots – a pole or zero at the origin: phase

- A pole at the origin gives rise to the term $-j\omega$.
  - $H(j\omega) = -j\omega = \omega e^{-j\frac{\pi}{2}} \Rightarrow \angle H(j\omega) = -\angle j\omega = -90^{o}$.
  - The phase is constant for all values of ω.

- A zero at the origin gives rise to the term $j\omega$.
  - $\angle H(j\omega) = \angle j\omega = 90^{o}$. The phase plot for a zero at the origin is a mirror image about the ω axis of the phase plot for a pole at the origin.



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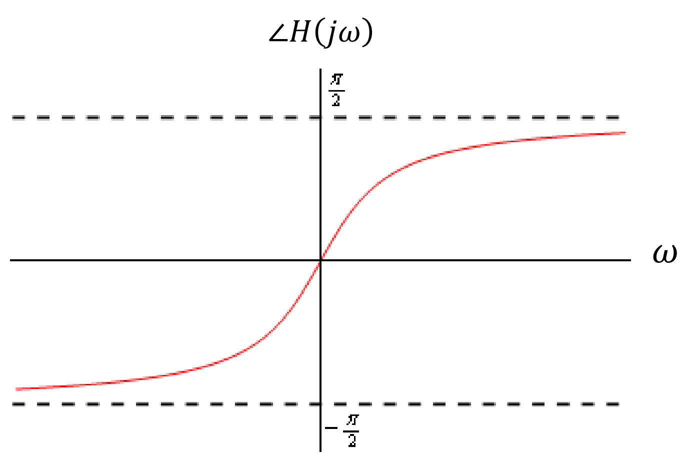
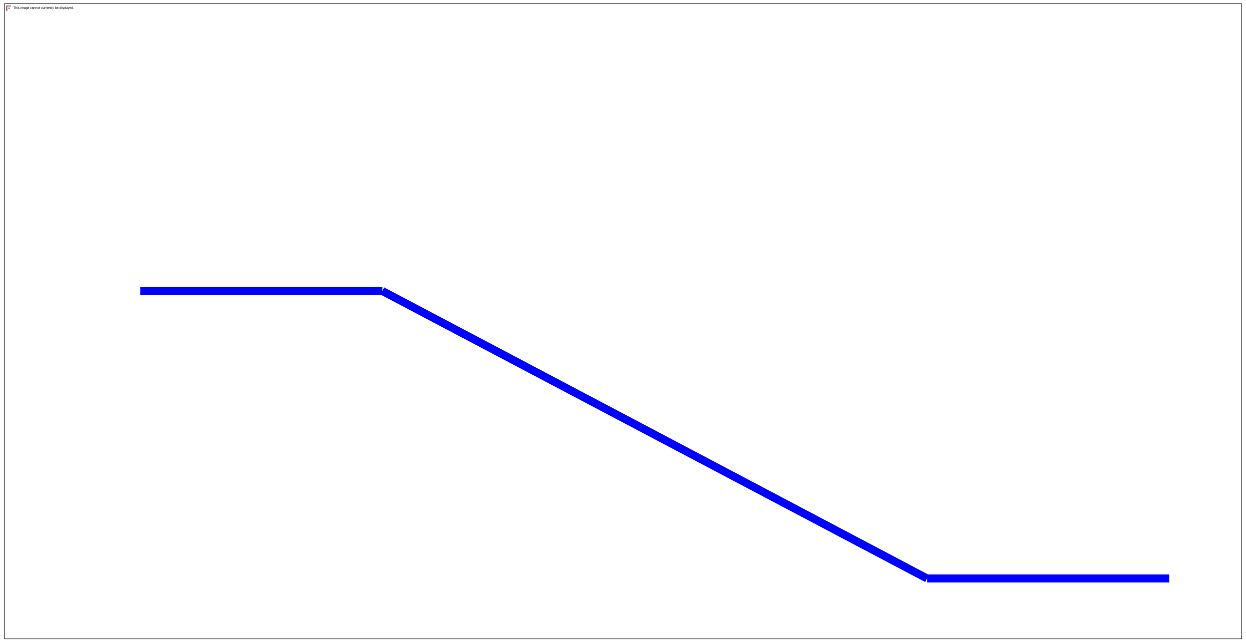
Bode plots – a first order pole or zero: phase

- A pole at $-a$ gives rise to the term $1+\frac{j\omega}{a}$.
  - $\angle H(j\omega) = -\angle\left(1+\frac{j\omega}{a}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$.
  - $\omega \ll a \Rightarrow \frac{\omega}{a} \ll 1 \Rightarrow -\tan^{-1}\left(\frac{\omega}{a}\right) \approx 0$
  - $\omega \gg a \Rightarrow \frac{\omega}{a} \gg 1 \Rightarrow -\tan^{-1}\left(\frac{\omega}{a}\right) \approx -90^{o}$
- The phase plot for a zero at $-a$ is a mirror image about the ω axis of the phase plot for a pole at the origin.



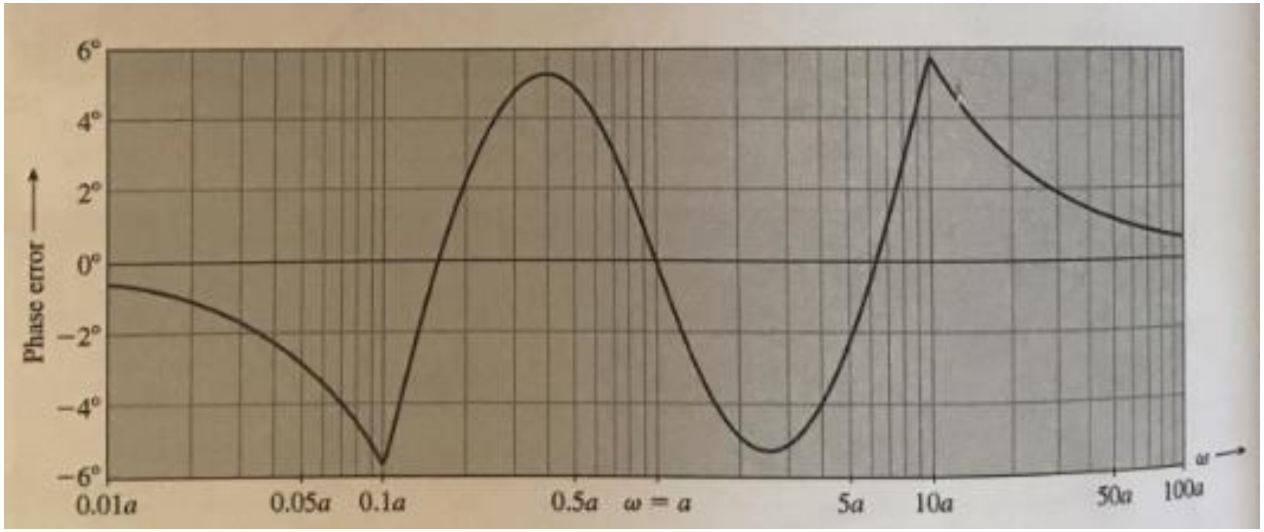
Bode plots – a first order pole or zero: phase

- We use a three-line segment asymptotic plot for greater accuracy. The asymptotes are:
  - $\omega \leq a/10 \Rightarrow 0^{\mathrm{o}}$
  - $\omega \geq 10a \Rightarrow -90^{\mathrm{o}}$
  - A straight line with slope -45^{o} /decade connects the above two asymptotes (from $\omega = a/10$ to $\omega = 10a$) crossing the ω axis at $\omega = a/10$.



Bode plots – a first order pole or zero: phase error

- The maximum error occurs at $\omega = 0.1a$ and $\omega = 10a$ and is 5.7°.
- The actual phase can be obtained if we add the error to the asymptotic plot.



$$\mathrm{err}(10a) = \mathrm{exact\_phase}(10a) - \mathrm{asymptotic\_phase}(10a)$$

$$= -\tan^{-1}\left(\frac{10a}{a}\right) - (-90^{o}) = -\tan^{-1}\left(\frac{10a}{a}\right) + 90^{o} = -\tan^{-1}(10) + 90^{o} = -84.28940686 + 90^{o} = 5.7^{o}.$$

$$\mathrm{err}(0.1a) = \mathrm{exact\_phase}(0.1a) - \mathrm{asymptotic\_phase}(0.1a)$$

$$= -\tan^{-1}\left(\frac{0.1a}{a}\right) - 0 = -\tan^{-1}(0.1) = -5.7^{o}.$$

Bode plots – second order complex conjugate poles : phase

- Now consider the term:

$$1 + 2j\zeta\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2 = 1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\,2\zeta\left(\frac{\omega}{\omega_n}\right)$$

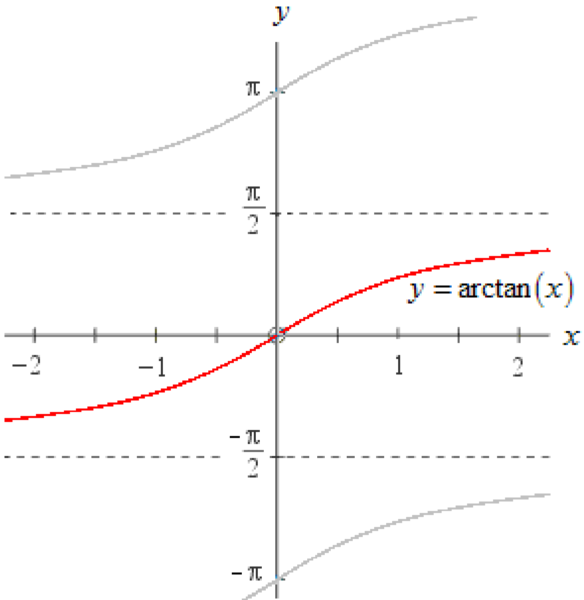
$$\angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

  - $\omega \ll \omega_n \Rightarrow \frac{\omega}{\omega_n} \approx 0,$

    $$\angle H(j\omega) = -\tan^{-1}\left(\frac{0}{1-0}\right) = -\tan^{-1}(0) = 0$$

  - $\omega \gg \omega_n \Rightarrow \frac{\omega}{\omega_n} \approx \infty,$

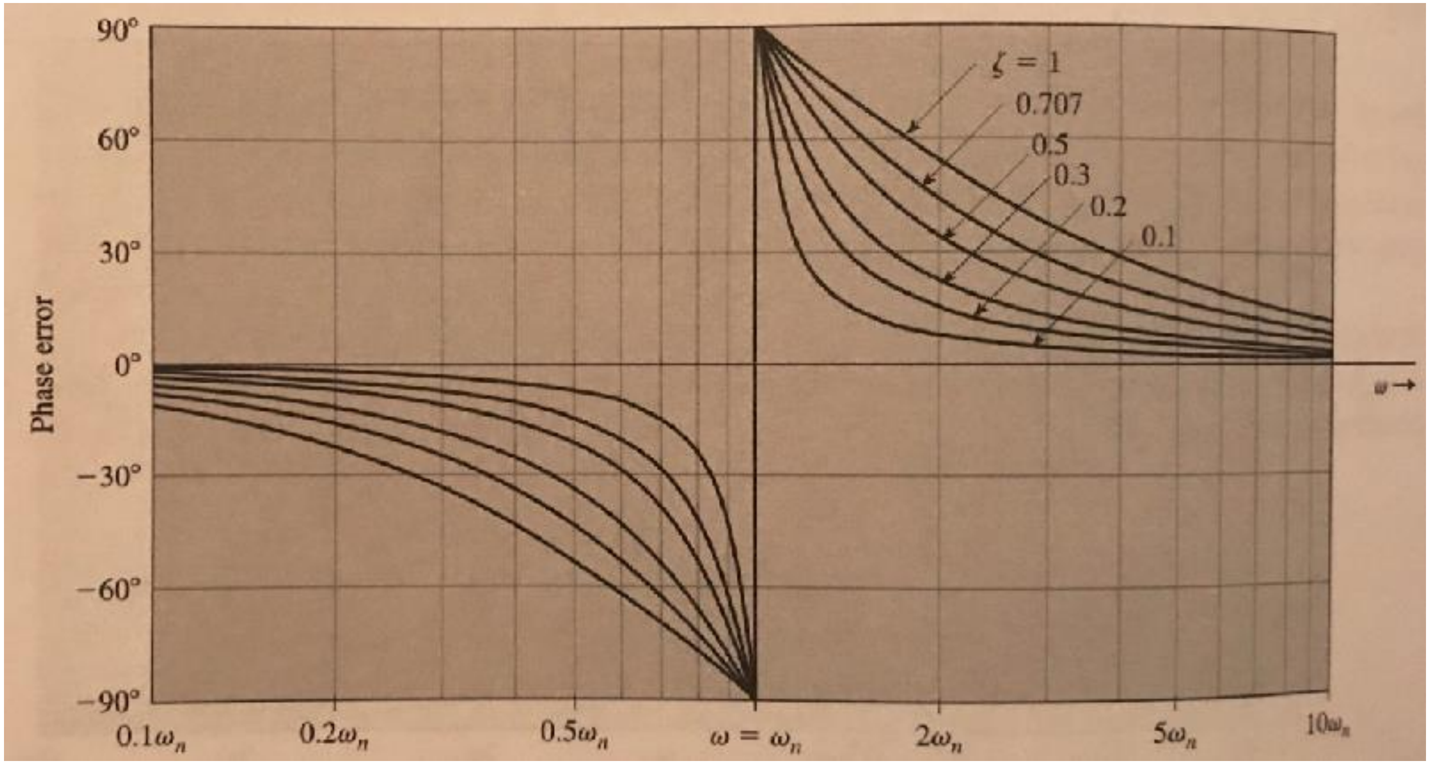
    $$\angle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta\infty}{1-\infty^2}\right) = -\tan^{-1}(0) = -180^{\circ}$$



- The phase involves a parameter ζ, resulting in a different plot for each value of ζ.

Bode plots – second order complex conjugate poles : phase error

- An error plot is shown in the figure below for various values of ζ.
- The actual phase can be obtained if we add the error to the asymptotic plot.



Bode plots example: phase

- Consider the previous system with transfer function:

$$H(s) = \frac{20s(s+100)}{(s+2)(s+10)} = 100\,\frac{s(1+\frac{s}{100})}{(1+\frac{s}{2})(1+\frac{s}{10})}$$

- For the pole at $s = -2$ ($a = -2$) the phase plot is:
  - $\omega \leq \frac{2}{10} = 0.2 \Rightarrow 0^o$
  - $\omega \geq 10 \cdot 2 = 20 \Rightarrow -90^o$
  - A straight line with slope -45^o /decade connects the above two asymptotes (from $\omega = 0.2$ to $\omega = 20$) crossing the ω axis at $\omega = 0.2$.
- For the pole at $s = -10$ ($a = -10$) the phase plot is:
  - $\omega \leq \frac{10}{10} = 1 \Rightarrow 0^o$
  - $\omega \geq 10 \cdot 10 = 100 \Rightarrow -90^o$
  - A straight line with slope -45^o /decade connects the above two asymptotes (from $\omega = 1$ to $\omega = 100$) crossing the ω axis at $\omega = 1$.

Bode plots example: phase cont.

- Consider the previous system with transfer function:

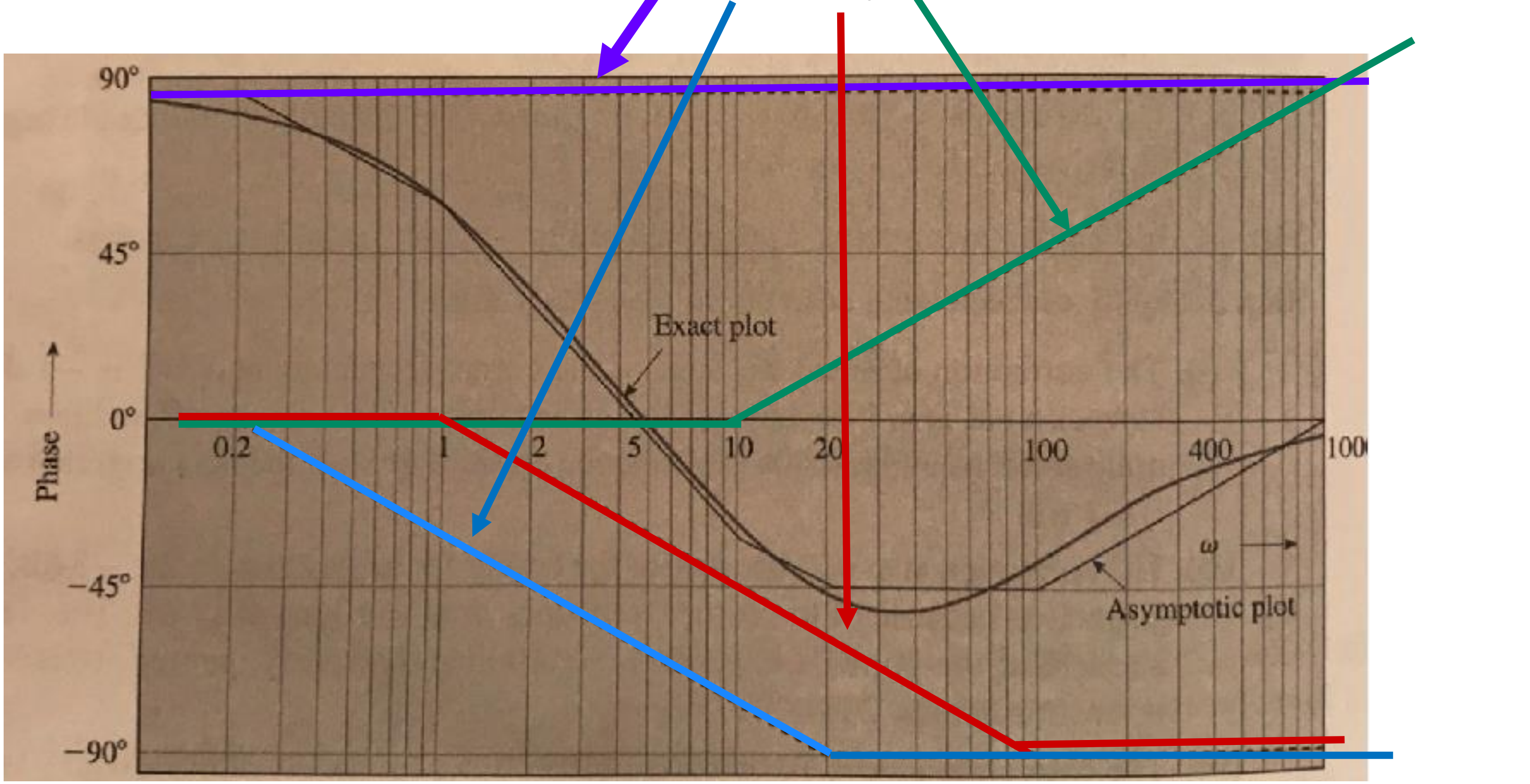
$$H(s) = \frac{20s(s+100)}{(s+2)(s+10)} = 100\,\frac{s(1+\frac{s}{100})}{(1+\frac{s}{2})(1+\frac{s}{10})}$$

- The zero at the origin causes a 90^{o} phase shift.
- For the zero at $s = -100$ ($a = -100$) the phase plot is:
  - $\omega \leq \frac{100}{10} = 10 \Rightarrow 0^{o}$
  - $\omega \geq 10 \cdot 100 = 1000 \Rightarrow 90^{o}$
  - A straight line with slope 45^{o}/decade connects the above two asymptotes (from $\omega = 10$ to $\omega = 1000$) crossing the ω axis at $\omega = 10$.

Bode plots example: total phase cont.

- Consider the previous system with transfer function:

$$H(s) = 100\,\frac{s\left(1+\frac{s}{100}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)}$$



Relating this lecture to other courses

- You will be applying frequency response in various areas such as filters and 2nd year control. You have also used frequency response in the 2nd year analogue electronics course. Here we explore this as a special case of the general concept of complex frequency, where the real part is zero.

- You have come across Bode plots from 2nd year analogue electronics course. Here we go deeper into where all these rules come from.

- We will apply much of what we have done so far in the frequency domain to analyse and design some filters in the next lecture.