

Signals and Systems

Problem Set 1

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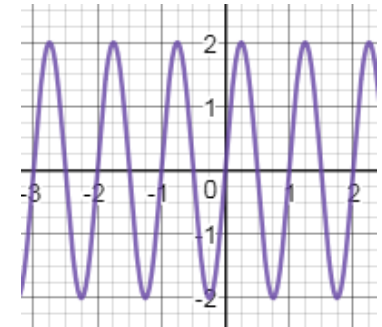
READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

Problem 1

Sketch each of the following continuous-time signals, specify if the signal is periodic/non-periodic, odd/even. If the signal is periodic specify its period.

(i) $x(t) = 2\sin(2\pi t)$

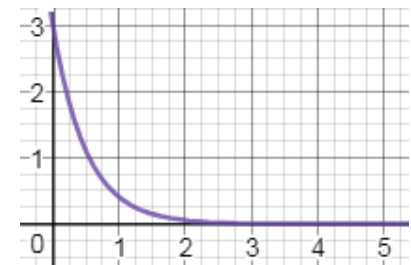
Periodic with period 1. **Odd** because $\sin(-t) = -\sin(t)$.



(ii) $x(t) = \begin{cases} 3e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$

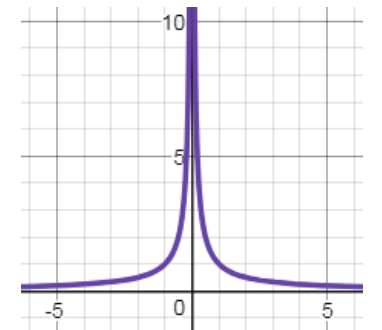
This is a **causal** system and therefore, it is **aperiodic**. A periodic system exists over the entire range of time from $-\infty$ to ∞ .

Neither odd nor even. An odd or even signal must have values within both the positive and the negative range.



(iii) $x(t) = \frac{1}{|t|}$

Aperiodic. **Even** because $x(-t) = x(t)$.



Problem 1 cont.

- For a signal of the form $x(t) = \sin(a\pi t) + \sin(b\pi t)$ to be periodic, the numbers a and b have to be **commensurable**.

In mathematics, two non-zero real numbers a and b are said to be commensurable if their ratio a/b is a **rational number**; otherwise a and b are called **incommensurable**. (Recall that a rational number is one that is equivalent to the ratio of two integers.)

Proof

We test the periodicity as follows:

$$x(t + T) = \sin(a\pi(t + T)) + \sin(b\pi(t + T)) = \sin(a\pi t) \cos(a\pi T) + \cos(a\pi t) \sin(a\pi T) + \sin(b\pi t) \cos(b\pi T) + \cos(b\pi t) \sin(b\pi T)$$

$x(t) = x(t + T)$ if $a\pi T = 2m\pi$ and $b\pi T = 2n\pi$, m, n integers. Therefore,

$$\frac{a}{b} = \frac{m}{n}$$

Hence, the numbers a and b have to be commensurable.

Problem 1 cont.

Based on the analysis of the previous slide we have:

$$(iv) \quad x(t) = \sin\left(\frac{2}{5}\pi t\right) + \sin\left(\frac{2}{3}\pi t\right) \quad a = \frac{2}{5}, \quad b = \frac{2}{3}, \quad \frac{a}{b} = \frac{3}{5} = \frac{m}{n}.$$

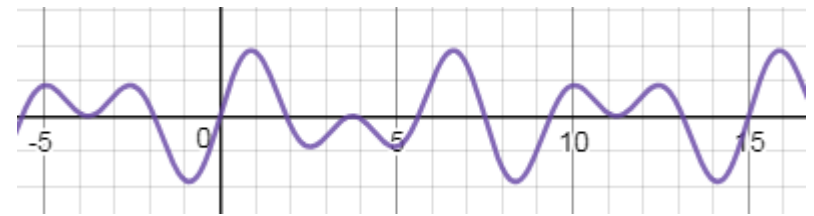
Hence, the numbers a and b are commensurable and the signal is **periodic**.

$$a\pi T = 2m\pi \Rightarrow T = \frac{2m}{a} = 5m$$

$$b\pi T = 2n\pi \Rightarrow T = \frac{2n}{b} = 3n.$$

Therefore $5m = 3n \Rightarrow m = 3, n = 5, T = 15$.

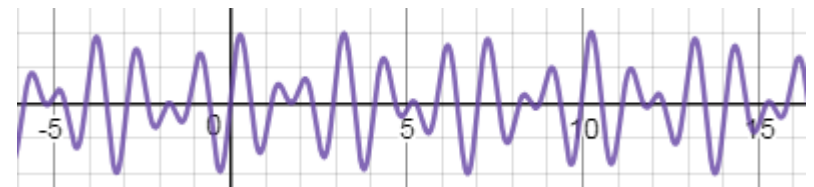
Furthermore, it is **odd**.



$$(v) \quad x(t) = \sin(2\pi t) + \sin(\sqrt{2}\pi t) \quad a = 2, \quad b = \sqrt{2}, \quad \frac{a}{b} = \sqrt{2} \neq \frac{m}{n}.$$

Hence, the numbers a and b are not commensurable and the signal is **aperiodic** (although it doesn't look like in the figure!).

Furthermore, it is **odd**.



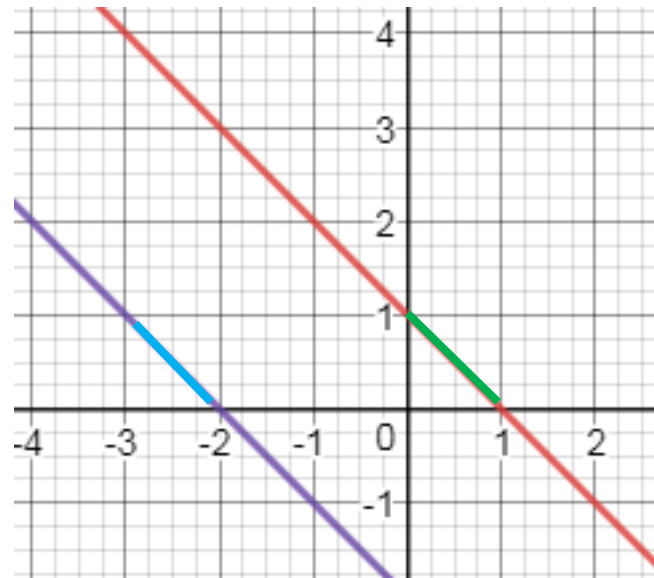
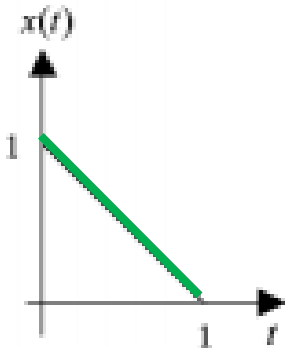
Problem 2

Sketch the signal:

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now sketch each of the following and describe briefly in words how each of the signals can be derived from the original signal $x(t)$.

(i) $x(t + 3) = \begin{cases} 1 - (t + 3) = -t - 2 & 0 \leq t + 3 \leq 1 \Rightarrow -3 \leq t \leq -2 \\ 0 & \text{otherwise} \end{cases}$ _____

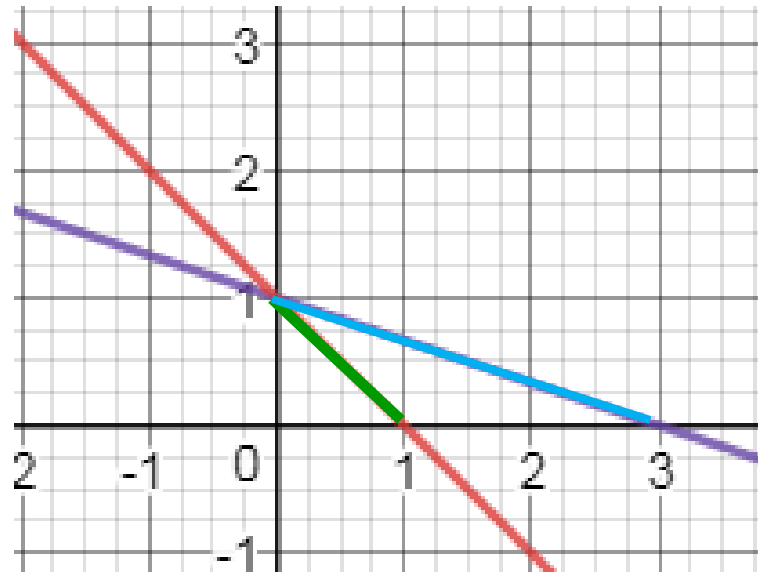


Problem 2 cont.

Sketch the signal:

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(ii) \quad x(t/3) = \begin{cases} 1 - t/3 & 0 \leq t/3 \leq 1 \Rightarrow 0 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$



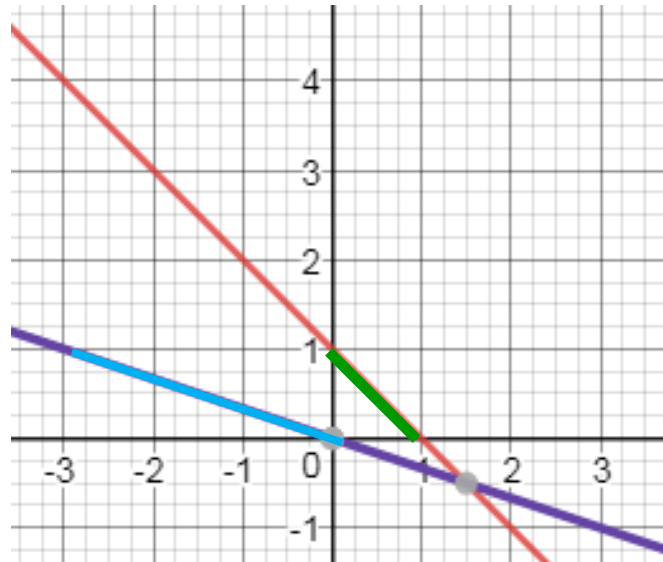
Problem 2 cont.

Sketch the signal:

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(iii)

$$x\left(\frac{t}{3} + 1\right) = \begin{cases} 1 - \left(\frac{t}{3} + 1\right) = -\frac{t}{3} & 0 \leq \frac{t}{3} + 1 \leq 1 \Rightarrow -1 \leq \frac{t}{3} \leq 0 \Rightarrow -3 \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



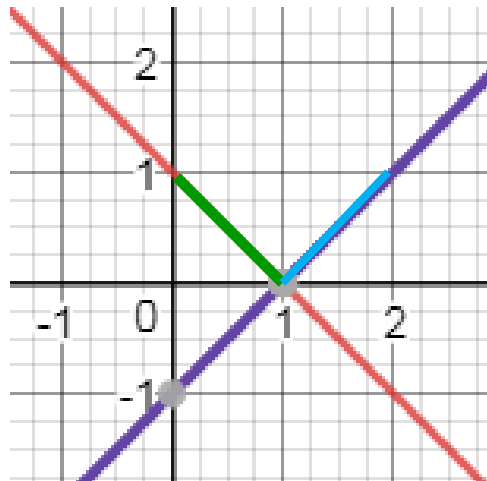
Problem 2 cont.

Sketch the signal:

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(iv)

$$\begin{cases} x(-t + 2) = \\ \begin{cases} 1 - (-t + 2) = t - 1 & 0 \leq -t + 2 \leq 1 \Rightarrow -1 \leq t - 2 \leq 0 \Rightarrow 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$



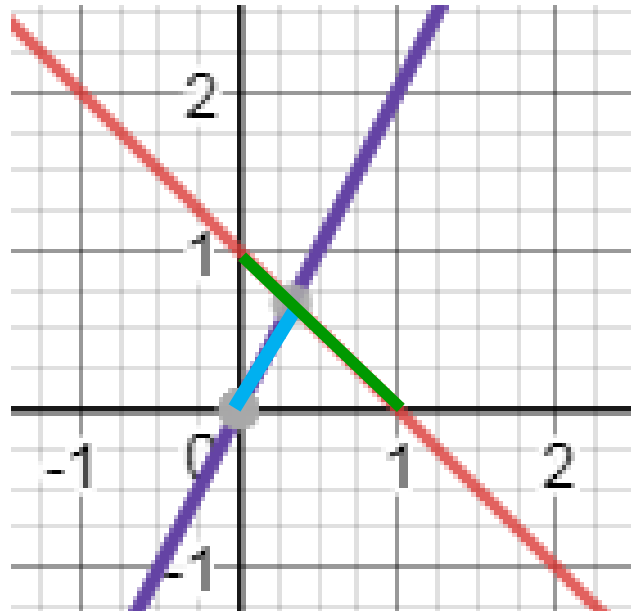
Problem 2 cont.

Sketch the signal:

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(v)

$$x(-2t + 1) = \begin{cases} 1 - (-2t + 1) = 2t & 0 \leq -2t + 1 \leq 1 \Rightarrow -1 \leq -2t \leq 0 \Rightarrow 0 \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



Problem 3

Sketch each of the following discrete-time signals, specify if the signal is periodic/non-periodic, odd/even. If the signal is periodic specify its period.

(i) $x[n] = \cos(n\pi)$

Periodic and **even**. $\cos(n + T)\pi = \cos(n\pi)\cos(T\pi) - \sin(n\pi)\sin(T\pi)$

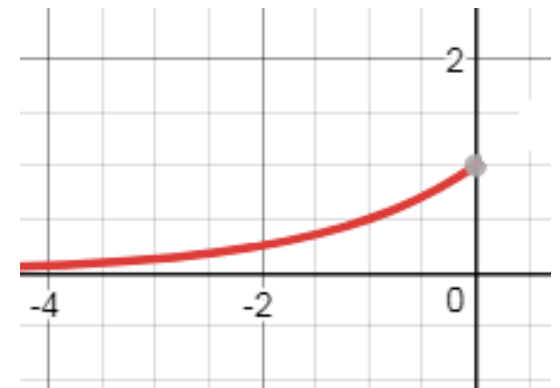
$$\cos(T\pi) = 1, \sin(T\pi) = 0 \Rightarrow \mathbf{T = 2}.$$

Signal form is obvious.

(ii) $x[n] = \begin{cases} 0.5^{-n} & n \leq 0 \\ 0 & n > 0 \end{cases}$

Aperiodic. Neither odd nor even.

The continuous version is depicted on the right.



Problem 4

Consider the rectangular function:

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0 & \text{otherwise} \end{cases}$$

(i) Sketch $x(t) = \sum_{k=0}^1 \Pi(t - k)$

$$\Pi(t - 1) = \begin{cases} 1 & |t - 1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < t - 1 < \frac{1}{2} \Rightarrow \frac{1}{2} < t < \frac{3}{2} \\ 1/2 & t = \frac{1}{2}, \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \sum_{k=0}^1 \Pi(t - k) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{3}{2} \\ 1/2 & t = -\frac{1}{2}, \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

(ii) Sketch $x(t) = \sum_{-\infty}^{\infty} \Pi(t - k)$. It is very easy to spot that $x(t) = 1$.

Problem 5

Consider a discrete-time signal $x[n]$, fed as input into a system. The system produces the discrete-time output $y[n]$ such that

$$y[n] = \begin{cases} x[n] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

- (i) Is the system described above memoryless? Explain
It is **memoryless** since the output at time instant n depends on the input only at time instant n and not past or future time instants.
- (ii) Is the system described above causal? Explain.
It is **causal** since the output at time instant n depends on the input only at time instant n and not future time instants.
- (iii) Are causal systems in general memoryless? Explain.
No. If the output at time instant n depends on the input at time instant n **and** past time instants the system is causal but not memoryless.

Problem 5 cont.

(iv) Is the system linear and time-invariant? Explain.

The output can be written in a compact form as a function of the input as:

$$y[n] = \frac{x[n] + (-1)^n x[n]}{2}$$

We see that if the input signal $x_1[n]$ produces an output signal $y_1[n]$ and the input signal $x_2[n]$ produces an output signal $y_2[n]$ then the input signal $a_1 x_1[n] + a_2 x_2[n]$ produces the output

$$y_3[n] = \frac{(a_1 x_1[n] + a_2 x_2[n]) + (-1)^n (a_1 x_1[n] + a_2 x_2[n])}{2} = a_1 y_1[n] + a_2 y_2[n].$$

Therefore, the system is **linear**.

However, if the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n - n_o]$ produces the output $y_1[n] = \frac{x[n - n_o] + (-1)^n x[n - n_o]}{2}$.

We see that $y[n - n_o] = \frac{x[n - n_o] + (-1)^{n - n_o} x[n - n_o]}{2} \neq y_1[n]$

Therefore, the system is **time varying**.

Problem 6

State with a brief explanation if the following systems are linear/non-linear, causal/non-causal, time invariant/time-varying.

(i) $y[n] = x[n] - x[n - 1]$

We see that if the input signal $x_1[n]$ produces an output signal $y_1[n]$ and the input signal $x_2[n]$ produces an output signal $y_2[n]$ then the input signal $a_1x_1[n] + a_2x_2[n]$ produces the output

$$y_3[n] =$$

$$a_1x_1[n] + a_2x_2[n] - a_1x_1[n - 1] - a_2x_2[n - 1] = a_1y_1[n] + a_2y_2[n]$$

Therefore, the system is **linear**.

If the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n - n_o]$ produces the output $y_1[n] = x[n - n_o] - x[n - 1 - n_o]$

We see that $y[n - n_o] = y_1[n]$

Therefore, the system is **time invariant**.

The system is **causal** since the output does not depend on future inputs.

Problem 6 cont.

(ii) $y[n] = \text{sgn}(x[n])$

We see that if the input signal $x_1[n]$ produces an output signal $y_1[n]$ and the input signal $x_2[n]$ produces an output signal $y_2[n]$ then the input signal $a_1x_1[n] + a_2x_2[n]$ produces the output

$$y_3[n] = \text{sgn}(a_1x_1[n] + a_2x_2[n]) \neq a_1y_1[n] + a_2y_2[n]$$

Therefore, the system is **non-linear**.

However, if the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n - n_o]$ produces the output $y_1[n] = \text{sgn}(x[n - n_o])$.

We see that $y[n - n_o] = \text{sgn}(x[n - n_o]) = y_1[n]$.

Therefore, the system is **time invariant**.

The system is **causal** since the output does not depend on future inputs.

Problem 6 cont.

(iii) $y[n] = n^2 x[n + 2]$

The input $x_1[n]$ produces an output $y_1[n]$.

The input $x_2[n]$ produces an output $y_2[n]$.

The input $a_1 x_1[n] + a_2 x_2[n]$ produces the output

$$y_3[n] = n^2 (a_1 x_1[n + 2] + a_2 x_2[n + 2]) = a_1 y_1[n] + a_2 y_2[n]$$

Therefore, the system is **linear**.

However, if the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n - n_o]$ produces the output $y_1[n] = n^2 x[n - n_o + 2]$.

We see that $y[n - n_o] = (n - n_o)^2 x[n - n_o + 2] \neq y_1[n]$

Therefore, the system is **time varying**.

The system is **non-causal** since if $n > 0$ then $n + 2 > n$ which shows that the output requires future values of the input in order to be calculated.

Problem 7

State with a brief explanation if the following systems are linear/non-linear, causal/non-causal, time invariant/time-varying. $x(t)$ is the input and $y(t)$ is the output.

(i) $y(t) = x(t)\cos(2\pi f_0 t + \phi)$

The input $x_1(t)$ produces the output $y_1(t) = x_1(t)\cos(2\pi f_0 t + \phi)$.

The input $x_2(t)$ produces the output $y_2(t) = x_2(t)\cos(2\pi f_0 t + \phi)$.

The linear combination $a_1 x_1(t) + a_2 x_2(t)$ produces the output $y_3(t) = (a_1 x_1(t) + a_2 x_2(t))\cos(2\pi f_0 t + \phi) = a_1 y_1(t) + a_2 y_2(t)$

Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_0)$ produces the output $y_1(t) = x(t - t_0)\cos(2\pi f_0 t + \phi)$

We see that $y(t - t_0) = x(t - t_0)\cos(2\pi f_0(t - t_0) + \phi) \neq y_1(t)$

Therefore, the system is **time-varying**.

The system is **causal** since the output does not depend on future inputs.

Problem 7 cont.

(ii) $y(t) = A\cos(2\pi f_0 t + x(t))$

The input $x_1(t)$ produces the output $y_1(t) = A\cos(2\pi f_0 t + x_1(t))$.

The input $x_2(t)$ produces the output $y_2(t) = A\cos(2\pi f_0 t + x_2(t))$.

The linear combination $a_1 x_1(t) + a_2 x_2(t)$ produces the output
 $y_3(t) = A\cos(2\pi f_0 t + a_1 x_1(t) + a_2 x_2(t)) \neq a_1 y_1(t) + a_2 y_2(t)$

Therefore, the system is **non-linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = A\cos(2\pi f_0 t + x(t - t_o))$.

We see that $y(t - t_o) = A\cos(2\pi f_0(t - t_o) + x(t - t_o)) \neq y_1(t)$

Therefore, the system is **time-varying**.

The system is **causal** since the output does not depend on future inputs.

Problem 7 cont.

(iii) $y(t) = \int_{-\infty}^t x(\delta)d\delta$

The input $x_1(t)$ produces the output $y_1(t) = \int_{-\infty}^t x_1(\delta)d\delta$.

The input $x_2(t)$ produces the output $y_2(t) = \int_{-\infty}^t x_2(\delta)d\delta$.

The linear combination $a_1x_1(t) + a_2x_2(t)$ produces the output

$$y_3(t) = \int_{-\infty}^t (a_1x_1(\delta) + a_2x_2(\delta))d\delta = a_1y_1(t) + a_2y_2(t)$$

Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$

produces the output $y_1(t) = \int_{-\infty}^t x_2(\delta - t_o)d\delta$.

Replace $\delta - t_o = \tau \Rightarrow y_1(t) = \int_{-\infty}^{t-t_o} x_2(\tau)d\tau$

We see that $y(t - t_o) = \int_{-\infty}^{t-t_o} x(\delta)d\delta = y_1(t)$

Therefore, the system is **time-invariant**.

The system is **causal** since the output does not depend on future inputs.

Problem 7 cont.

(iv) $y(t) = x(2t)$

The input $x_1(t)$ produces the output $y_1(t) = x_1(2t)$.

The input $x_2(t)$ produces the output $y_2(t) = x_2(2t)$.

The linear combination $a_1x_1(t) + a_2x_2(t)$ produces the output

$$y_3(t) = a_1x_1(2t) + a_2x_2(2t) = a_1y_1(t) + a_2y_2(t)$$

Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = x(2t - t_o)$.

We see that $y(t - t_o) = x(2(t - t_o)) \neq y_1(t)$

Therefore, the system is **time-varying**.

The system is **non-causal** since if $t > 0$ then $2t > t$ which shows that the output requires future values of the input in order to be calculated.

Problem 7 cont.

(iv) $y(t) = x(-t)$

The input $x_1(t)$ produces the output $y_1(t) = x_1(-t)$.

The input $x_2(t)$ produces the output $y_2(t) = x_2(-t)$.

The linear combination $a_1x_1(t) + a_2x_2(t)$ produces the output

$$y_3(t) = a_1x_1(-t) + a_2x_2(-t) = a_1y_1(t) + a_2y_2(t)$$

Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = x(-t - t_o)$.

We see that $y(t - t_o) = x(t - t_o) \neq y_1(t)$.

Therefore, the system is **time varying**.

The system is **non-causal** since if $t < 0$ then $-t > t$ which shows that the output requires future values of the input in order to be calculated.