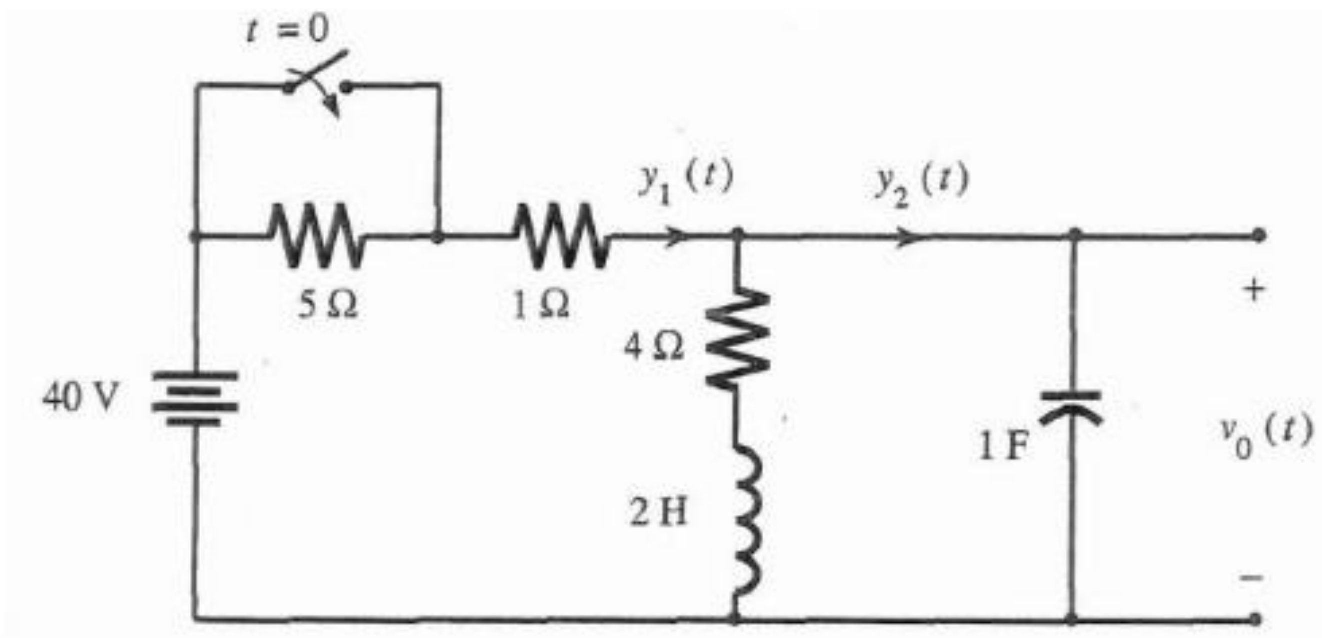
Problem Sheet 5 Question 4

- **Problem:** For the circuit shown in the figure below, the switch is in open position for a long time before $t = 0$, when it is closed instantaneously.



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a) Let us examine what happens for $t < 0$.

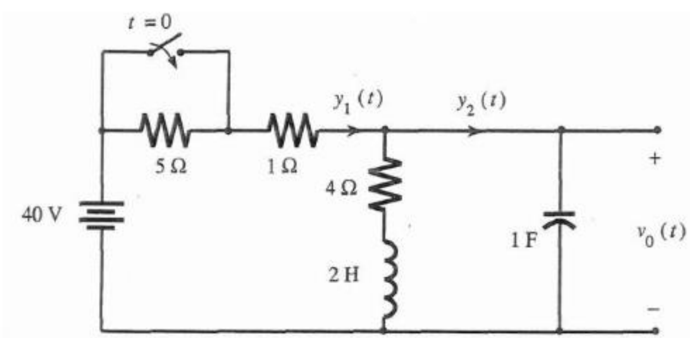
We know that $i_C(t) = C\frac{dv_C(t)}{dt}$ (or $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau$) and $v_L(t) = L\frac{di_L(t)}{dt}$.

- If the system is in steady state, the current $y_2(t)$ across the capacitor is 0 since the voltage across the capacitor is constant. Therefore, $y_2(0^-) = 0\ A$.
- Furthermore, the current $y_1(t)$ which flows through the left loop (Loop 1) is constant and therefore, the voltage across the inductor is 0.
- Based on the above two points, the voltage across the capacitor $v_0(t)$ is the same as the voltage across the $4\ \Omega$ resistor, i.e.,$y_1(t) \cdot 4\ V$.

In that case for the left loop (Loop 1) we have:

$y_1(t) \cdot (5 + 1 + 4) = 40V \Rightarrow y_1(t) = 4\ A, t < 0$

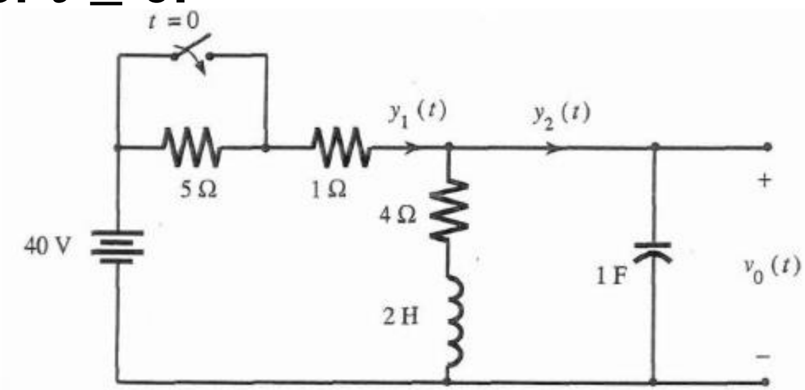
$v_0(t) = y_1(t) \cdot 4\ V = 16V, t < 0$



Initial conditions: $y_1(0^-) = 4\ A, y_2(0^-) = 0\ A,\ v_0(0^-) = 16\ V$

Problem Sheet 5 Question 4 cont.

a) **Write loop equations in time domain for $t \geq 0$.**



Loop 1:

$$L\frac{di_L(t)}{dt} + 4 \cdot i_L(t) + 1 \cdot y_1\ (t) = 40\ u(t)$$

$$i_L(t) = y_1(t) - y_2(t)$$

$$L(\frac{dy_1(t)}{dt} - \frac{dy_2(t)}{dt}) + 4(y_1(t) - y_2\ (t)) + 1 \cdot y_1\ (t) = 40\ u(t)$$

$$2\frac{dy_1(t)}{dt} - 2\frac{dy_2(t)}{dt} + 5y_1(t) - 4y_2\ (t) = 40u(t)$$

Loop 2:

$$L\left(\frac{dy_1(t)}{dt} - \frac{dy_2(t)}{dt}\right) + 4(y_1(t) - y_2\ (t)) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\, d\tau = \frac{1}{C}\int_{-\infty}^{t} y_2(\tau)\, d\tau$$

Problem Sheet 5 Question 4 cont.

b) Loop 1 equation in time and Laplace domain for $t \geq 0$.

Loop 1:

$$2\frac{dy_1(t)}{dt} - 2\frac{dy_2(t)}{dt} + 5y_1(t) - 4y_2(t) = 40\,u(t)$$

$$y_1(0^-) = 4\ A$$

$$y_2(0^-) = 0\ A$$

$$2[sY_1(s) - y_1(0^-)] - 2[sY_2(s) - y_2(0^-)] + 5Y_1(s) - 4Y_2(s) = 40/s \Rightarrow$$

$$(2s + 5)Y_1(s) - (2s + 4)\,Y_2(s) = 8 + 40/s$$

Problem Sheet 5 Question 4 cont.

b) Loop 2 equation in time and Laplace domain for $t \geq 0$.

Loop 2:

$$L\left(\frac{dy_1(t)}{dt} - \frac{dy_2(t)}{dt}\right) + 4(y_1(t) - y_2(t)) = \int_{-\infty}^{t} y_2(\tau)\, d\tau$$

$$y_1(0^-) = 4\ A$$

$$y_2(0^-) = 0\ A$$

$$v_0(0^-) = 16\ V$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \Leftrightarrow \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\, dt$$

$$\mathcal{L}\left\{\int_{-\infty}^{t} y_2(\tau)\, d\tau\right\} = \frac{Y_2(s)}{s} + \frac{1}{s}\frac{1}{C}\int_{-\infty}^{0^-} y_2(t)\, dt = \frac{Y_2(s)}{s} + \frac{1}{s} v_0(0^-)$$

$$2[sY_1(s) - y_1(0^-)] - 2[sY_2(s) - y_2(0^-)] + 4Y_1(s) - 4Y_2(s) = \frac{Y_2(s)}{s} + \frac{16}{s}$$

$$-(2s + 4)Y_1(s) + \left(2s + 4 + \frac{1}{s}\right) Y_2(s) = -8 - \frac{16}{s}$$

Problem Sheet 5 Question 4 cont.

b) Merge equations for Loops 1 and 2 in a matrix form $t \geq 0$

$$\begin{bmatrix} 2s+5 & -(2s+4) \\ -(2s+4) & 2s+4+\frac{1}{s} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 8+\frac{40}{s} \\ -8-\frac{16}{s} \end{bmatrix}$$

By solving the above system we obtain:

$$Y_1(s) = \frac{4(6s^2+13s+5)}{s(s^2+3s+2.5)} = \frac{8}{s} + \frac{16s+28}{(s^2+3s+2.5)}$$

$$Y_2(s) = \frac{20(s+2)}{(s^2+3s+2.5)}$$

Problem Sheet 5 Question 4 cont.

b) Find $y_1(t)$, $t \geq 0$

$$Y_1(s) = \frac{4(6s^2+13s+5)}{s(s^2+3s+2.5)} = \frac{8}{s} + \frac{16s+28}{s^2+3s+2.5}$$

We use Property 10c from Laplace Properties tables.

$$re^{-at}\cos(bt+\theta)\,u(t) \Leftrightarrow \frac{As+B}{s^2+2as+c}$$

$$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$$

$$\theta = \tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$$

$$b = \sqrt{c-a^2}$$

$$A = 16, B = 28, a = 1.5, c = 2.5$$

$$y_1(t) = [8 + 17.89e^{-1.5t}\cos(0.5t - 26.56^o)]\,u(t)$$

Problem Sheet 5 Question 4 cont.

b) Find $y_1(t), t \geq 0$

$$Y_2(s) = \frac{20(s+2)}{(s^2+3s+2.5)} = \frac{20s+40}{(s^2+3s+2.5)}$$

Property 10c

$$re^{-at}\cos(bt+\theta)\,u(t) \Leftrightarrow \frac{As+B}{s^2+2as+c}$$

$$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$$

$$\theta = tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$$

$$b = \sqrt{c-a^2}$$

$$A = 20, B = 40, a = 1.5, c = 2.5$$

$$y_2(t) = 20\sqrt{2}e^{-1.5t}\cos(0.5t-45^{o})\,u(t)$$