Imperial College London

Signals and Systems

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DR TANIA STATHAKI

READER (ASSOCIATE PROFFESOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

Signal transmission through LTI systems

- We have seen previously that if $x(t)$ and $y(t)$ are input and output of a LTI system with impulse response $h(t)$, then:

$$Y(\omega) = H(\omega)X(\omega)$$

- We can, therefore, perform LTI system analysis with Fourier transform in a way similar to that of Laplace transform.
- However, FT is more restrictive than Laplace transform because the system must be stable and $x(t)$ must itself be Fourier transformable.
- Laplace transform can be used to analyse stable and unstable systems, and applies to signals that grow exponentially.
- As already mentioned, if a system is stable, it can be shown that the frequency response of the system $H(j\omega)$ is just the Fourier transform of $h(t)$ (i.e., $H(\omega)$):

$$H(\omega) = H(s)\Big|_{s=j\omega}$$

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Time domain vs frequency domain

System's response to Dirac function (impulse response)

$$\delta(t) \implies h(t)$$

$x(t)$ as a sum of shifted impulse components

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$y(t)$ as a sum of responses to impulse components

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

System's response to $e^{j\omega t}$ is $H(\omega)e^{j\omega t}$

$$e^{j\omega t} \implies H(\omega)e^{j\omega t}$$

$x(t)$ as a sum of everlasting exponential components

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$y(t)$ as a sum of responses to exponential components

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}\,d\omega$$

Signal distortion during transmission

- In certain types of systems we require the input to pass through the system without distortion. For example:
  - Signal transmission over a communication channel.
  - Amplifying systems.

- Distortionless transmission implies that the output is the same as the input apart from:
  - A constant multiplicative factor.
  - A delay.

- Therefore, if $x(t)$ is the input and $y(t)$ is the output, distortionless transmission implies that:

$$y(t) = G_0 x(t - t_d)$$

Signal distortion during transmission cont.

- Distortionless transmission of an input $x(t)$ implies that:

$$y(t) = G_0 x(t - t_d)$$

- Taking the Fourier transform of the above yields:

$$Y(\omega) = G_0 X(\omega) e^{-j\omega t_d}$$

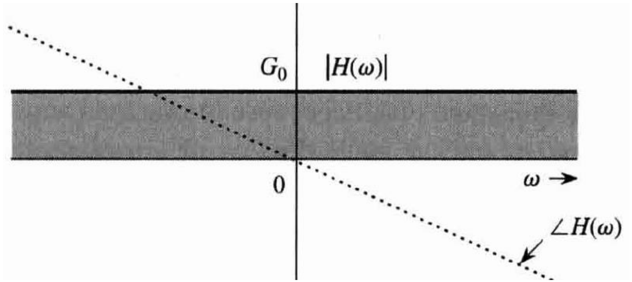
- Knowing that:

$$Y(\omega) = H(\omega) X(\omega)$$

we can write that the transfer function of a distortionless system is:

$$H(\omega) = G_0 e^{-j\omega t_d}$$

  - $|H(\omega)| = G_0$ amplitude response must be a constant
  - $\angle H(\omega) = -\omega t_d$ phase response must be a linear function of ω with slope $-t_d$ which also passes through the origin



Group delay

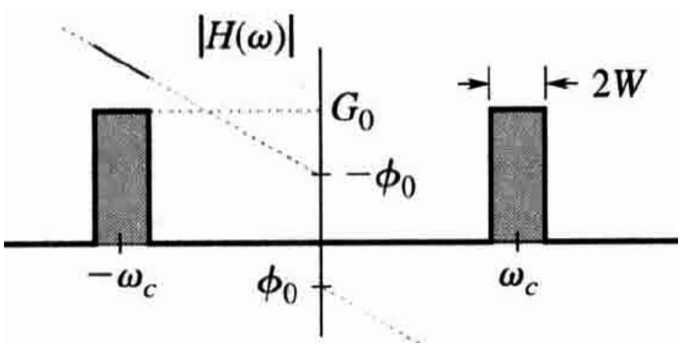
- In order to assess phase linearity we can find the slope of $\angle H(\omega)$ as a function of frequency and see whether it is constant. We define:

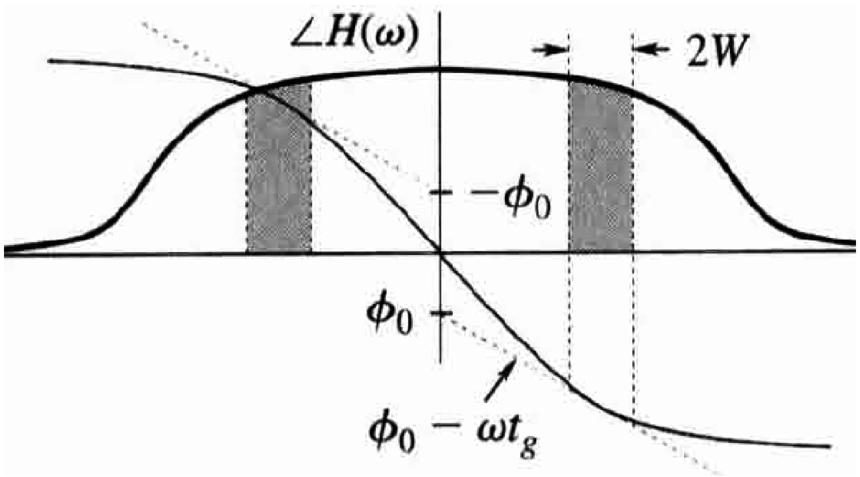
$$t_g(\omega) = -\frac{d}{d\omega}\angle H(\omega)$$

- $t_g(\omega)$ is called group delay or envelope delay.
- Note that a phase response given by $\angle H(\omega) = \phi_0 - \omega t_d$ also has a constant group delay. From now on we can write $t_d = t_g$.
- Therefore, the condition for phase linearity by testing whether the group delay is constant is more relaxed.
- Human ears are sensitive to amplitude distortion, but not phase distortion.
- Human eyes are sensitive to phase distortion, but not so much to amplitude distortion (recall the experiment where we have combined the amplitude of one image and the phase of another).

Bandpass systems and group delay

- For lowpass systems, the phase must be linear over the band of interest and also must pass through the origin.
- For bandpass systems, the phase must be linear over the band of interest but does not have to pass through the origin.
- Consider the following bandpass LTI system.





- The pass band is of width $2W$ centred at ω_c.

Bandpass systems and group delay cont.

- Within the pass band and for $\omega \geq 0$ the phase can be described as

$$\angle H(\omega) = \phi_0 - \omega t_g$$

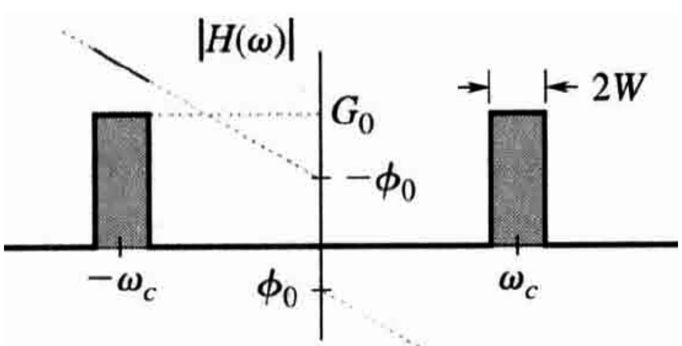
- The phase is always an odd function, and therefore,

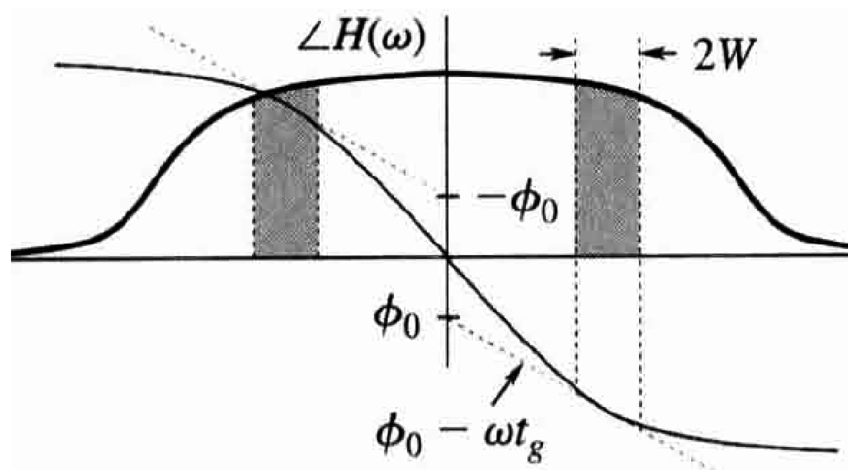
$$\angle H(-\omega) = -\angle H(\omega) = -(\phi_0 - \omega t_g) = -\phi_0 + \omega t_g$$

- We can write:

$$\angle H(\omega) = \begin{cases} \phi_0 - \omega t_g & \omega \geq 0 \\ -\phi_0 - \omega t_g & \omega < 0 \end{cases}$$

- For a distortionless system we have $H(\omega) = G_o e^{j(\phi_0 - \omega t_g)}$, $\omega \geq 0$.





Bandpass systems and group delay cont.

- Consider the distortionless system $H(\omega) = G_o e^{j(\phi_0 - \omega t_g)}$, $\omega \geq 0$.
- Consider the bandpass modulated signal $z(t) = x(t)\cos\omega_c t$ centred at ω_c where $x(t)$ is a lowpass signal with bandwidth W.
  - $\cos\omega_c t$ is the carrier of $z(t)$
  - $x(t)$ is the envelope of $z(t)$
- Consider now the input $\hat{z}(t) = x(t)e^{j\omega_c t}$ with $\hat{Z}(\omega) = X(\omega - \omega_c)$.
- The corresponding output is:

$$\hat{Y}(\omega) = H(\omega)\,\hat{Z}(\omega) = H(\omega)X(\omega - \omega_c)$$

$$\hat{Y}(\omega) = G_0 X(\omega - \omega_c)e^{j(\phi_0 - \omega t_g)} = G_0 e^{j\phi_0} X(\omega - \omega_c)e^{-j\omega t_g}$$

- We use the properties:
  - If $x(t) \Leftrightarrow X(\omega)$ then:

    $x(t - t_0) \Leftrightarrow X(\omega)e^{-j\omega t_0}$ and $x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega - \omega_0)$.
- We obtain: $\hat{y}(t) = G_0 e^{j\phi_0} x\left(t - t_g\right) e^{j\omega_c (t - t_g)} = G_0 x\left(t - t_g\right) e^{j[\omega_c (t - t_g) + \phi_0]}$

Bandpass systems and group delay cont.

- Consider the distortionless system $H(\omega) = G_o e^{j(\phi_0 - \omega t_g)}$, $\omega \geq 0$.
- We showed that for the input $\hat{z}(t) = x(t)e^{j\omega_c t}$ the output is:

$$\hat{y}(t) = G_0 x(t - t_g) e^{j[\omega_c(t - t_g) + \phi_0]}$$

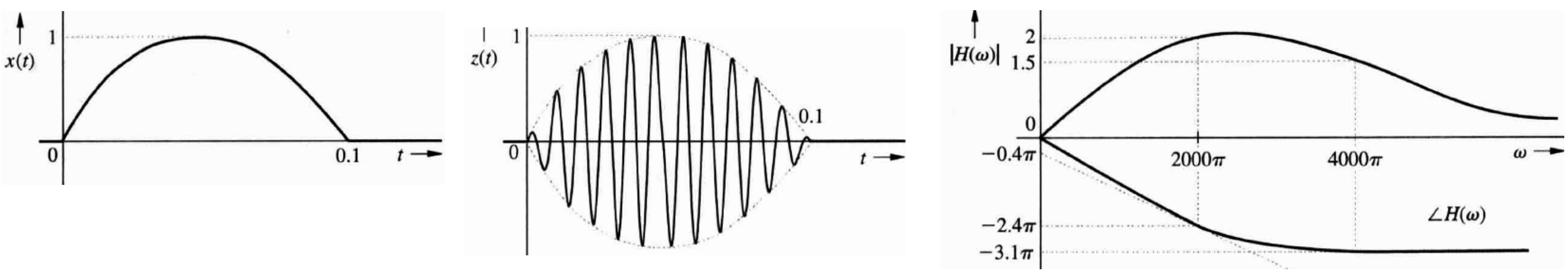
- For the input $z(t) = x(t)\cos\omega_c t = \mathrm{Re}\{\hat{z}(t)\}$ the output is

$$y(t) = \mathrm{Re}\{\hat{y}(t)\} = \mathrm{Re}\left\{G_0 x(t - t_g) e^{j[\omega_c(t - t_g) + \phi_0]}\right\}$$
$$= G_0 x(t - t_g)\cos[\omega_c(t - t_g) + \phi_0]$$

  - The output envelope $x(t - t_g)$ remains undistorted.
  - The output carrier acquires an extra phase ϕ_0.
  - In a modulation system the transmission is considered distortionless if the envelope $x(t)$ remains undistorted. This is because the signal information is contained solely in the envelope.
  - Therefore, the above type of transmission is considered distortionless.

Example

- A signal $z(t)$ shown below is given by $x(t)\cos\omega_c t$ where $\omega_c = 2000\pi$. The pulse $x(t)$ is a lowpass pulse of duration 0.1sec and has a bandwidth of about $10Hz$. This signal is passed through a filter whose frequency response is shown below. Find and sketch the filter output $y(t)$.

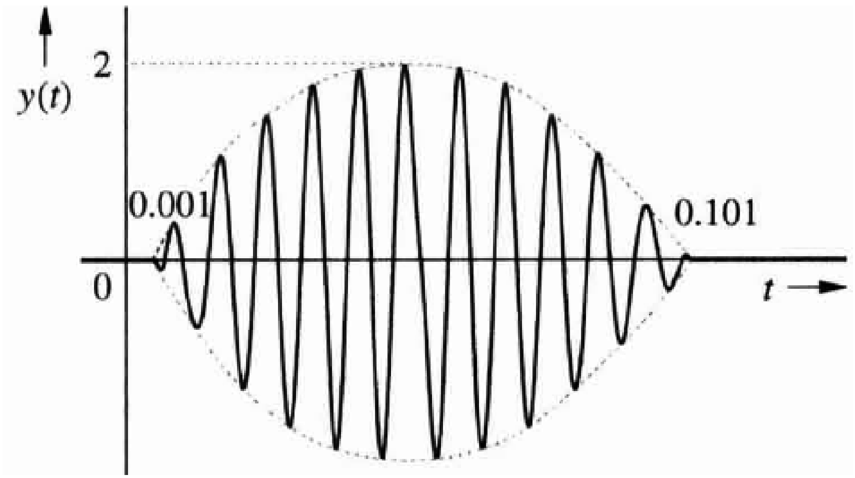


- $z(t)$ is a narrow band signal with bandwidth of $20Hz$ centred around $f_c = \omega_c/2\pi = 1kHz$.
- The gain at the centre frequency of $1kHz$ is 2.
- The group delay is: $t_g = \frac{2.4\pi - 0.4\pi}{2000\pi} = 10^{-3}$. It can be found by drawing the tangent at ω_c.
- The intercept along the vertical axis by the tangent is $\phi_0 = -0.4\pi$.

Example cont.

- Based on the above analysis the output of the system is:

$$y(t) = G_0 x(t - t_g)\cos[\omega_c(t - t_g) + \phi_0]$$
$$= 2x(t - 10^{-3})\cos[2000\pi(t - 10^{-3}) - 0.4\pi]$$



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Signal energy: Parseval's theorem

- The energy of a signal $x(t)$ can be derived either in time or in frequency domain:

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Proof

$$E_x = \int_{t=-\infty}^{\infty} x(t)x^*(t)dt = \int_{t=-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega\right]dt$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)\left[\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)X(\omega)d\omega = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega$$

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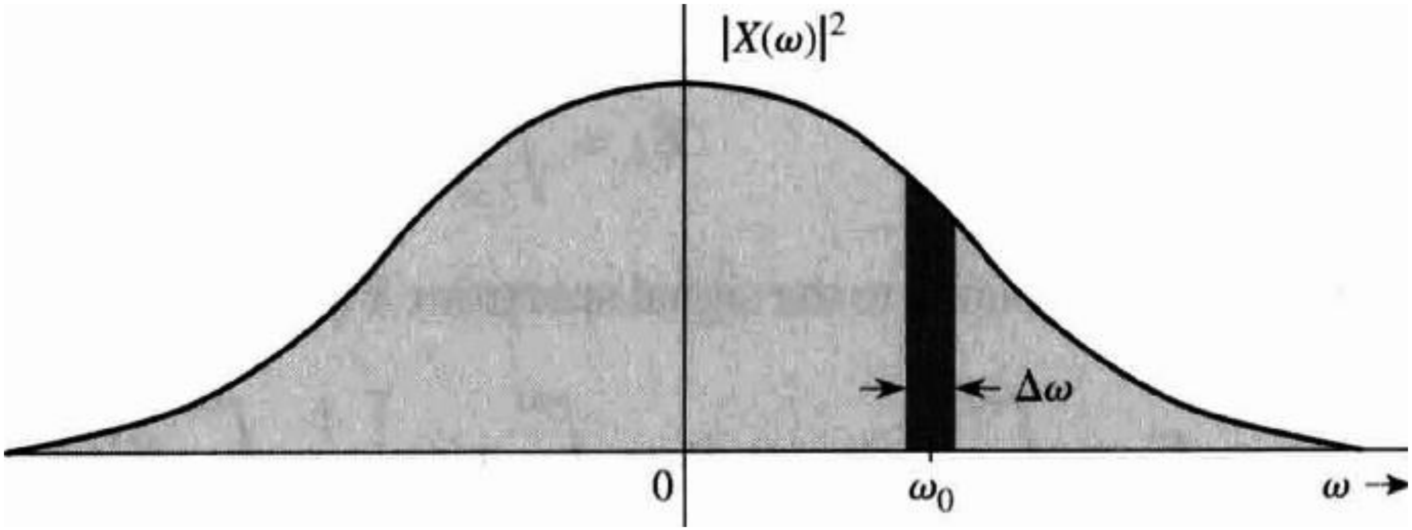
Signal energy: Parseval's theorem cont.

- The total energy of a signal is the area under the curve $|X(\omega)|^2$ divided by 2π.



- The energy over a small frequency band $\Delta\omega$, where $\Delta\omega \rightarrow 0$ is:

$$\Delta E_x = \frac{1}{2\pi}|X(\omega)|^2\,\Delta\omega = |X(\omega)|^2\,\Delta f,\ \Delta f = \frac{\Delta\omega}{2\pi}Hz$$

- The function $|X(\omega)|^2$ is the energy spectral density (per unit bandwidth in Hz).

Energy spectral density of a real signal

- If $x(t)$ is a real signal, then $X(\omega)$ and $X(-\omega)$ are conjugate.

- In that case $|X(\omega)|^2$ is even, since $|X(\omega)|^2 = X(\omega)X^*(\omega) = X(\omega)X(-\omega)$.

- Therefore,

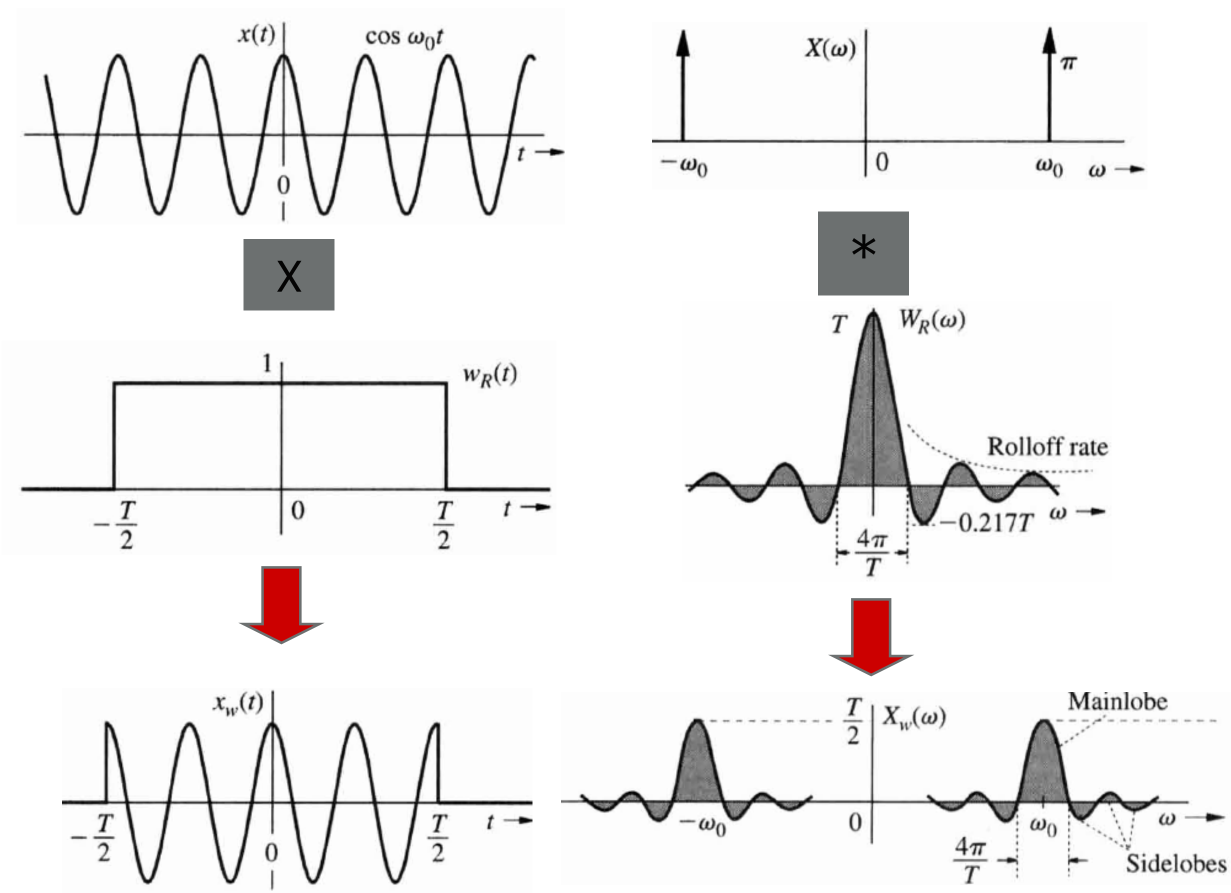
$$E_x = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}|X(\omega)|^2 d\omega = 2\frac{1}{2\pi}\int_{\omega=0}^{\infty}|X(\omega)|^2 d\omega$$
$$= \frac{1}{\pi}\int_{\omega=0}^{\infty}|X(\omega)|^2 d\omega$$

- Consequently, in a real signal the energy contributed by all spectral components between ω_1 and ω_2 is:

$$\Delta E_x = \frac{1}{\pi}\int_{\omega_1}^{\omega_2}|X(\omega)|^2 d\omega$$

Windowing and its effect

- Extracting a segment of a signal in time is the same as multiplying the signal with a rectangular window:



Spectral spreading

Energy spread out from ω_0 to width of $\approx 2\pi/T$.

Leakage

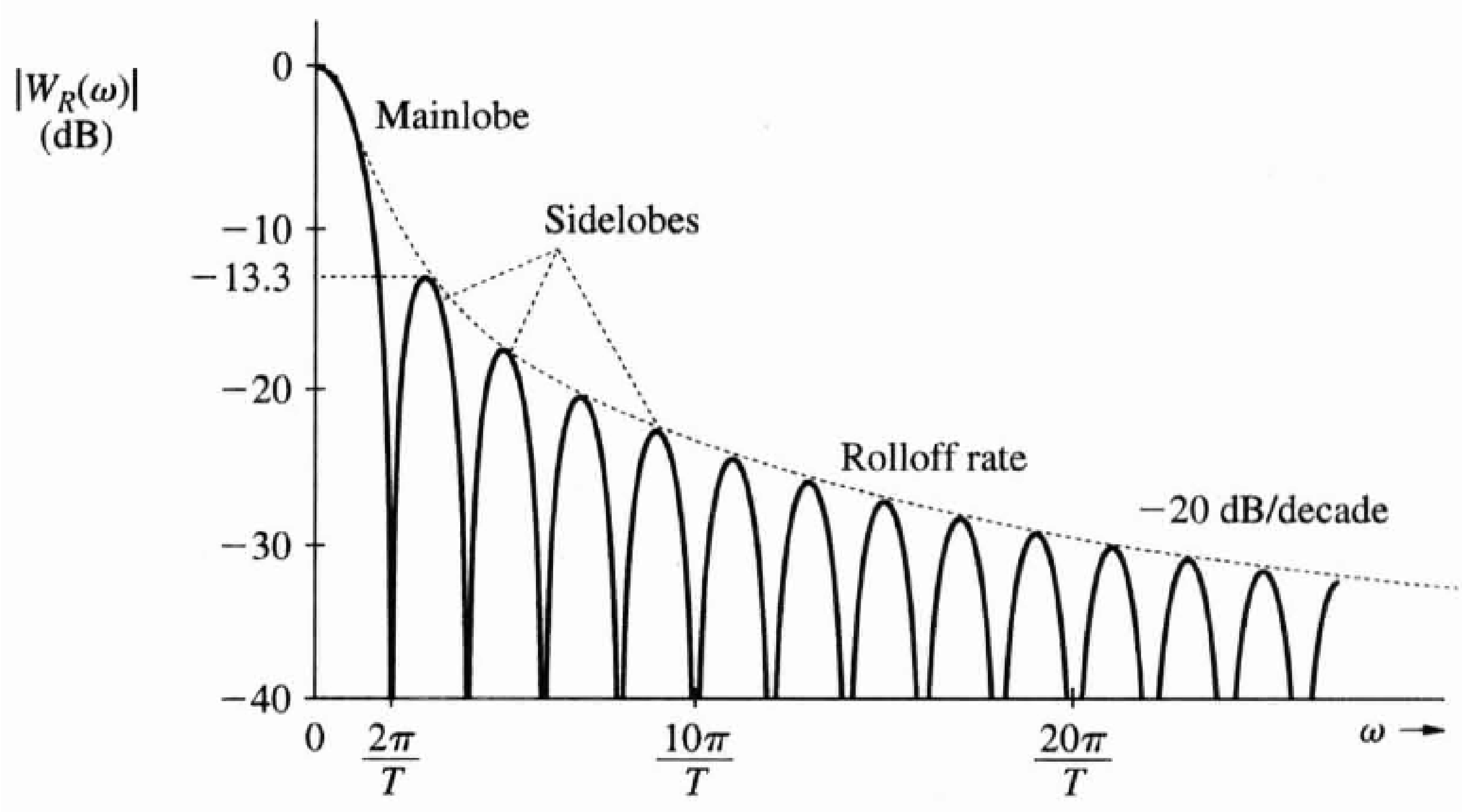
Energy leaks out from the mainlobe to the sidelobes.

Windowing and its effect cont.

- Using the previous example as a basis to understand windowing effects observe that:
  - If $x(t)$ has two spectral components of frequencies which differ by less than $\frac{4\pi}{T} rad/sec$ $(\frac{2}{T} Hz)$ they will be indistinguishable in the truncated signal.
  - The result is loss of spectral resolution.

Mainlobe and sidelobes of a rectangular window in dB

- Amplitude spectrum of a rectangular window in dB.

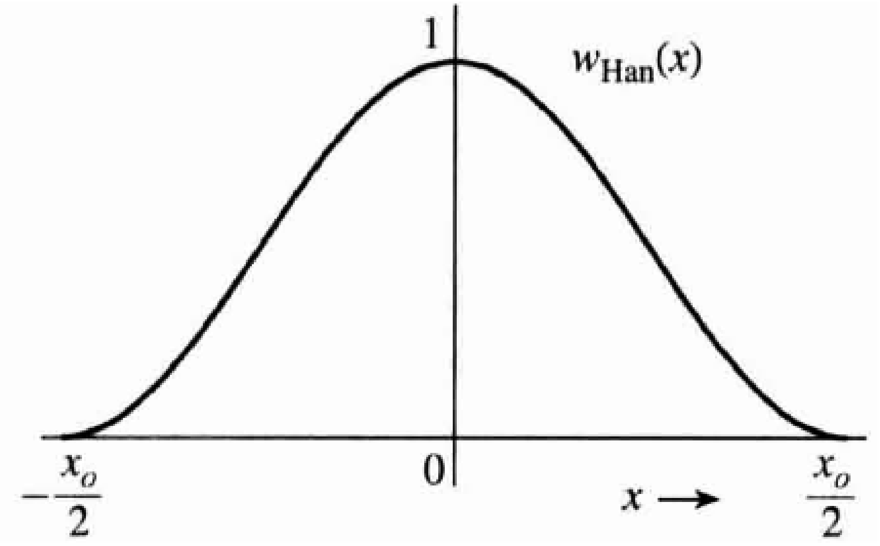


Remedies for side effects of truncation

1. Make mainlobe width as narrow as possible; this implies as wide a window as possible.
2. Avoid big discontinuity in the windowing function to reduce leakage (i.e., high frequency sidelobes).
3. 1) and 2) above are incompatible – therefore needs a compromise.

   Commonly used windows apart from the rectangular window are:

   - Hamming windows
   - Hanning windows
   - Barlett windows
   - Blackman windows
   - Kaiser windows



Comparison of different windowing functions

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	-6	-13.3
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	-12	-26.5
3	Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$	$\frac{8\pi}{T}$	-18	-31.5
4	Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	-6	-42.7
5	Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	-18	-58.1
6	Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $\quad 0 \le \alpha \le 10$	$\frac{11.2\pi}{T}$	-6	-59.9 ($\alpha = 8.168$)