

Signals and Systems

Lecture 11 Wednesday 22nd November 2017

DR TANIA STATHAKI

READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

Effect on poles and zeros on frequency response

- Consider a generic system transfer function

$$H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(s - z_1)(s - z_2) \dots (s - z_N)}{(s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_N)}$$

- The value of the transfer function at some complex frequency $s = p$ is:

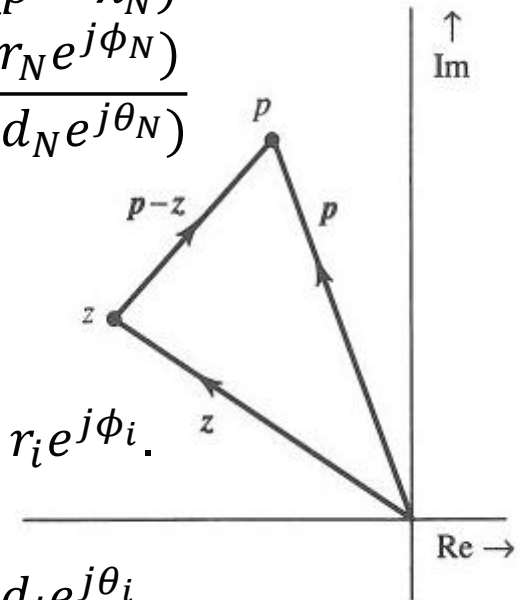
$$H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(p - z_1)(p - z_2) \dots (p - z_N)}{(p - \lambda_1)(p - \lambda_2) \dots (p - \lambda_N)}$$

$$H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2}) \dots (r_N e^{j\phi_N})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2}) \dots (d_N e^{j\theta_N})}$$

- The factor $p - z_i$ is a complex number.

- It is represented by a vector drawn from point z_i to point p in the complex plane.
- Using polar coordinates we can write $p - z_i = r_i e^{j\phi_i}$.
with $r_i = |p - z_i|$ and $\phi_i = \angle(p - z_i)$

- Same comments are valid for the factor $p - \lambda_i = d_i e^{j\theta_i}$.



Effect on poles and zeros on frequency response cont.

- The previous form can be further modified as:

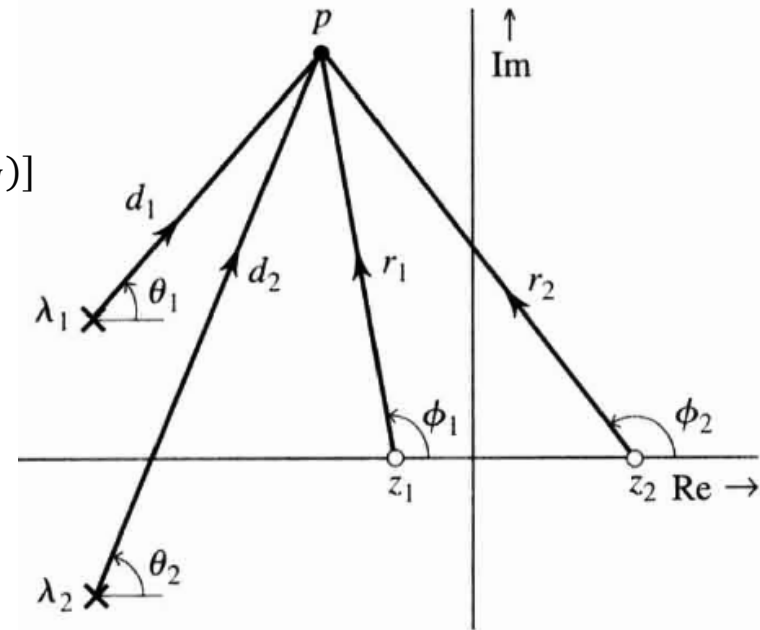
$$\begin{aligned}
 H(s) &= b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2}) \dots (r_N e^{j\phi_N})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2}) \dots (d_N e^{j\theta_N})} \\
 &= b_0 \frac{r_1 r_2 \dots r_N}{d_1 d_2 \dots d_N} e^{j[(\phi_1 + \phi_2 + \dots + \phi_N) - (\theta_1 + \theta_2 + \dots + \theta_N)]}
 \end{aligned}$$

- Therefore, the magnitude and phase at $s = p$ are given by:

$$\begin{aligned}
 |H(s)|_{s=p} &= b_0 \frac{r_1 r_2 \dots r_N}{d_1 d_2 \dots d_N} \\
 &= b_0 \frac{\text{product of the distances of zeros to } p}{\text{product of the distances of poles to } p}
 \end{aligned}$$

$$\begin{aligned}
 \angle H(s)_{s=p} &= (\phi_1 + \phi_2 + \dots + \phi_N) - (\theta_1 + \theta_2 + \dots + \theta_N) \\
 &= \text{sum of zeros' angles to } p - \text{sum of poles' angles to } p
 \end{aligned}$$

- If b_0 is negative, there is an additional phase π .

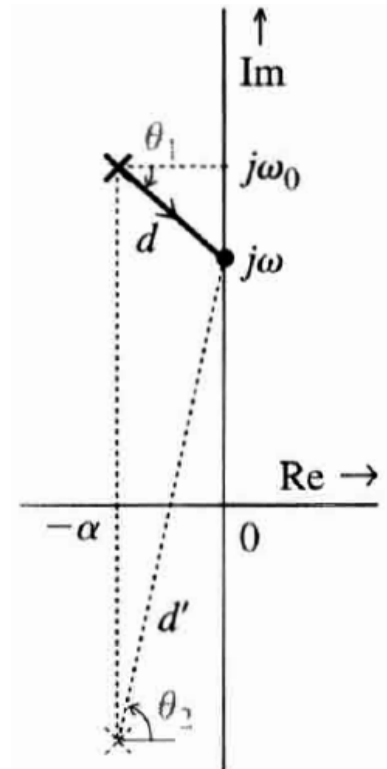


Gain enhancement by a single pole

- Consider the hypothetical case of a single pole at $-a + j\omega_0$.
- The amplitude response at a specific value of ω , $|H(j\omega)|$, is found by measuring the length of the line that connects the pole to the point $j\omega$.
- If the length of the above mentioned line is d , then $|H(j\omega)|$ is proportional to $\frac{1}{d}$.

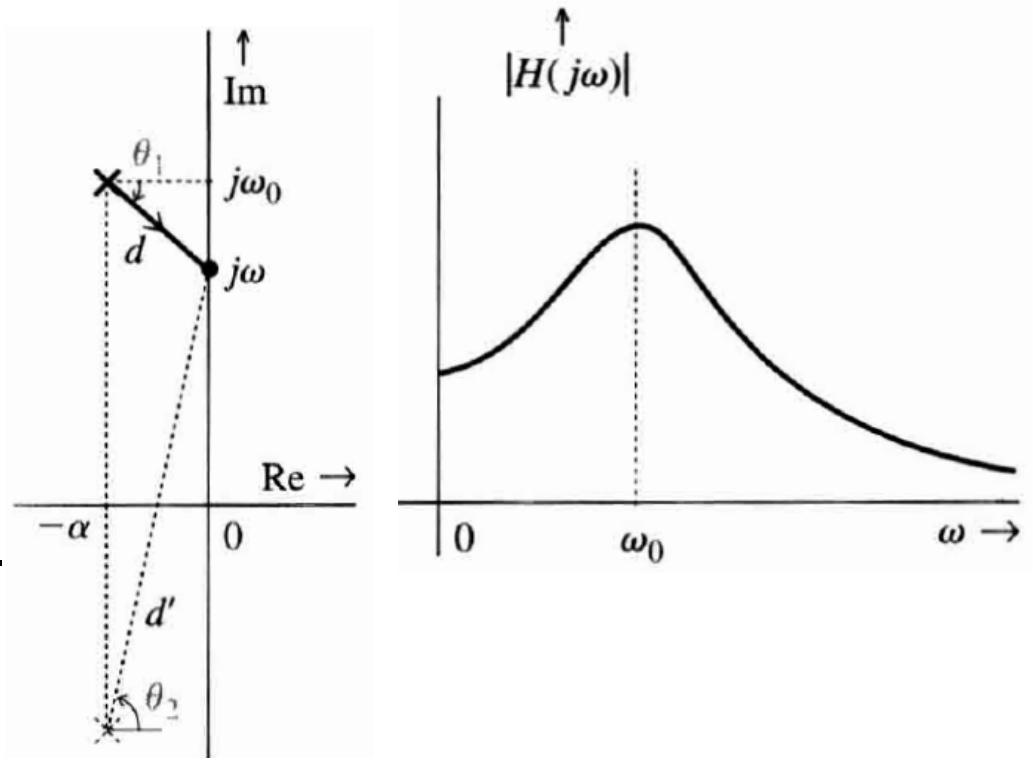
$$|H(j\omega)| = \frac{K}{d}$$

- As ω increases from zero, d decreases progressively until ω reaches the value ω_0 .
- As ω increases beyond ω_0 , d increases progressively.
- Therefore, the peak of $|H(j\omega)|$ occurs at ω_0 .
As a becomes smaller, i.e., as the pole moves closer to the imaginary axis the gain enhancement at ω_0 becomes more prominent (d becomes very small.)



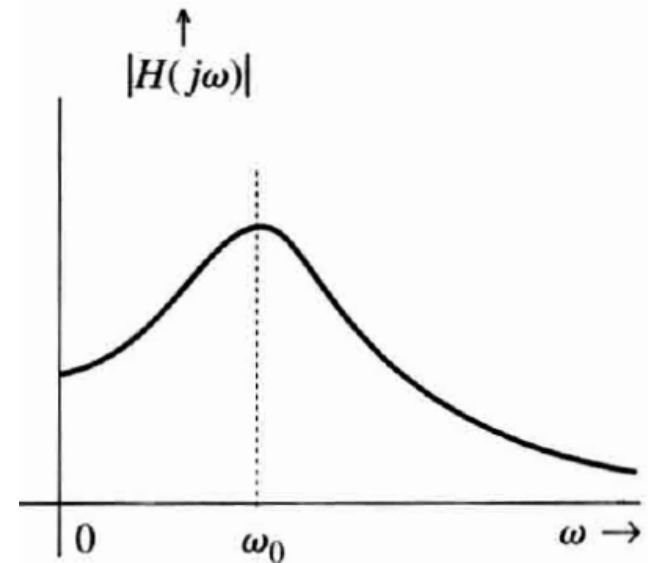
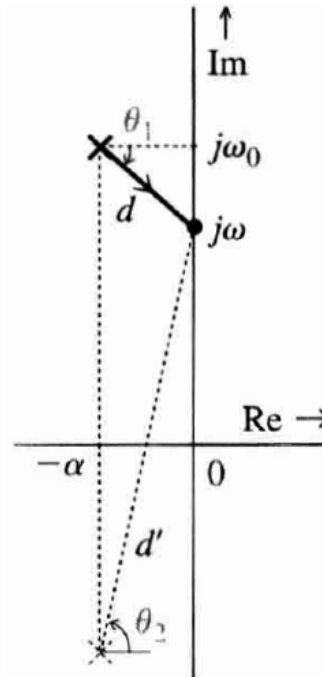
Gain enhancement by a single pole cont.

- In the extreme case of $a = 0$ (pole on the imaginary axis) the gain at ω_0 goes to infinity.
- Repeated poles further enhance the frequency selective effect.
- In conclusion, we can enhance a gain at a frequency ω_0 by placing a pole opposite the point $j\omega_0$.
- The closer the pole is to $j\omega_0$, the higher is the gain at ω_0 and furthermore, the enhancement is more prominent around ω_0 .
- Recall that poles must lie on the left half of the s -plane.



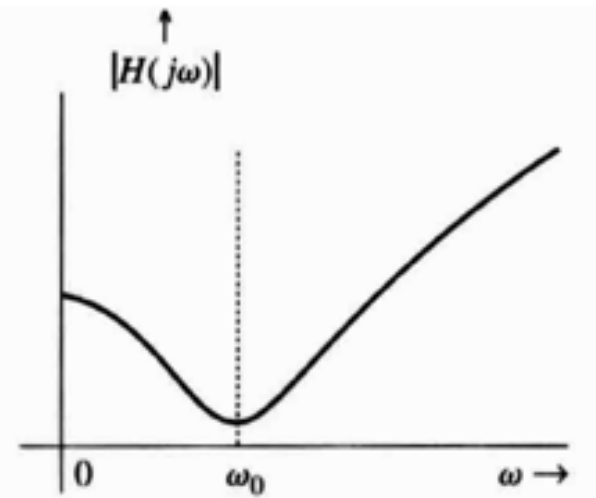
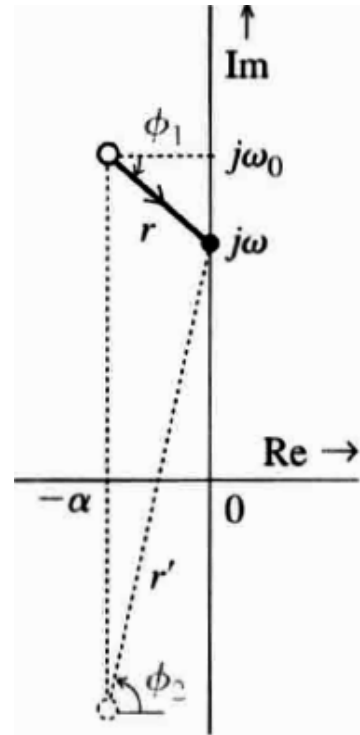
Gain enhancement by a pair of complex conjugate poles

- In a real system, a complex pole at $-a + j\omega_0$ must be accompanied by its conjugate pole $-a - j\omega_0$.
- The amplitude response at a specific value of ω , $|H(j\omega)|$, is found by measuring the length of the two lines that connect the poles to the point $j\omega$.
- If the lengths of the above mentioned lines are d , d' then $|H(j\omega)| = \frac{K}{dd'}$.
- We can see graphically that the presence of the conjugate pole does not affect substantially the behaviour of the system around ω_0 . This is because as we move around ω_0 , d' does not change dramatically.



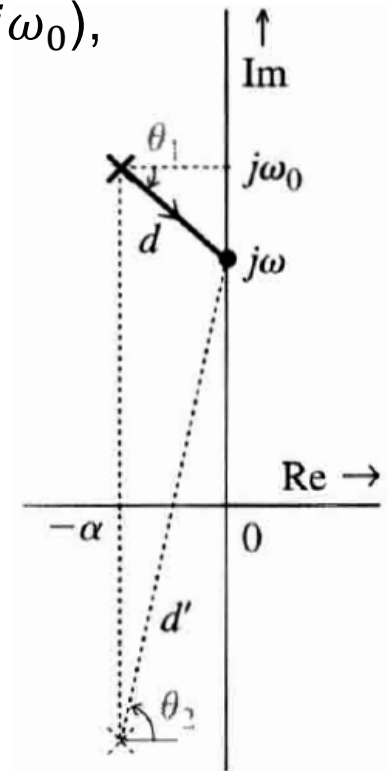
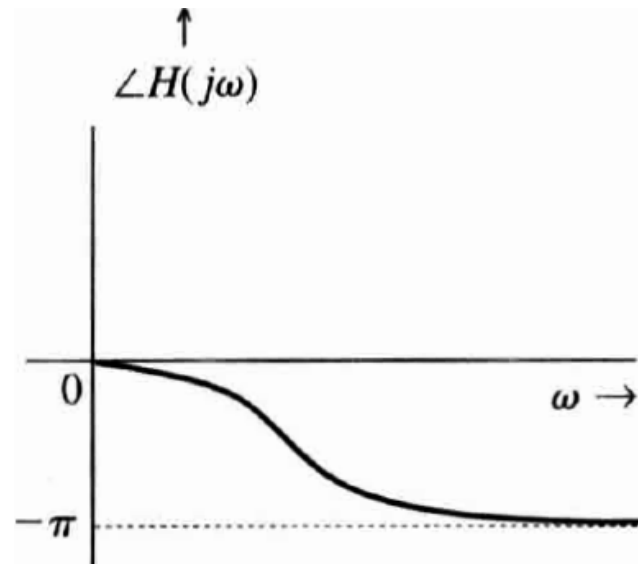
Gain suppression by a pair of complex conjugate zeros

- Consider a real system with a pair of complex conjugate zeros at $-a + j\omega_0$ and $-a - j\omega_0$.
- The amplitude response at a specific value of ω , $|H(j\omega)|$ is again found by measuring the length of the two lines that connect the zeros to the point $j\omega$.
- If the lengths of the above mentioned lines are r, r' then $|H(j\omega)| = Krr'$.
- In that case, the minimum of $|H(j\omega)|$ occurs at ω_0 .
- As a becomes smaller, i.e., as the zero moves closer to the imaginary axis, the gain suppression at ω_0 becomes more prominent.



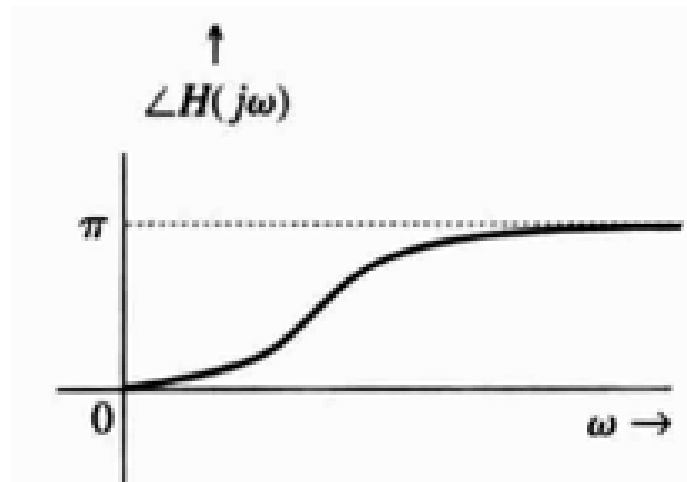
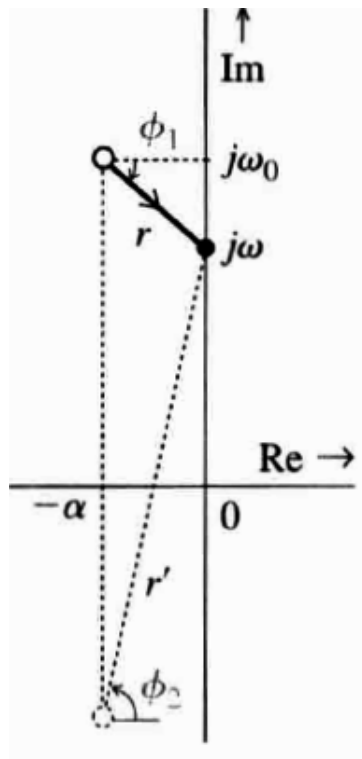
Phase response due to a pair of complex conjugate poles

- Angles formed by the poles $-a + j\omega_0$ and $-a - j\omega_0$ at $\omega = 0$ are equal and opposite.
- As ω increases from 0 up, the angle θ_1 (due to pole $-a + j\omega_0$), which has a negative value at $\omega = 0$, is reduced in magnitude.
- As ω increases from 0 up, the angle θ_2 (due to pole $-a - j\omega_0$), which has a positive value at $\omega = 0$, increases in magnitude.
- As a result, $\theta_1 + \theta_2$, the sum of the two angles, increases continuously and approaches a value of π as $\omega \rightarrow \infty$.
- $\angle H(j\omega) = -(\theta_1 + \theta_2)$



Phase response due to a pair of complex conjugate zeros

- Similar arguments regarding the phase are applied for a pair of complex conjugate zeros $-a + j\omega_0$ and $-a - j\omega_0$.
- $\angle H(j\omega) = (\phi_1 + \phi_2)$

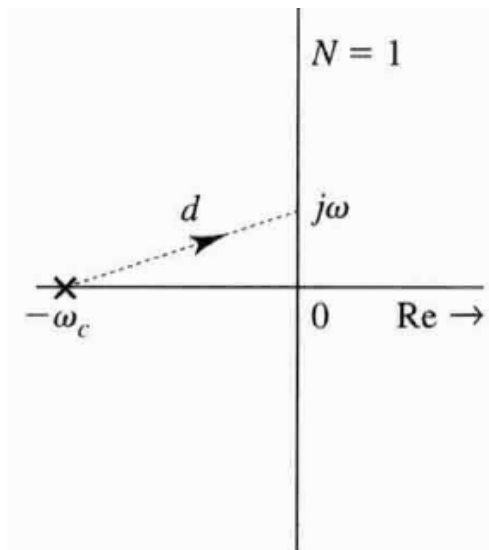


Lowpass filters. The simplest case.

- A lowpass filter is a system with a frequency response that has its maximum gain at $\omega = 0$.
- We showed in detail previously that a pole enhances the gain of the frequency response at frequencies which are within its close neighbourhood.
- Therefore, for a maximum gain at $\omega = 0$, we must place pole(s) on the real axis, within the left half plane, opposite the point $\omega = 0$.
- The simplest lowpass filter can be described by the transfer function:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

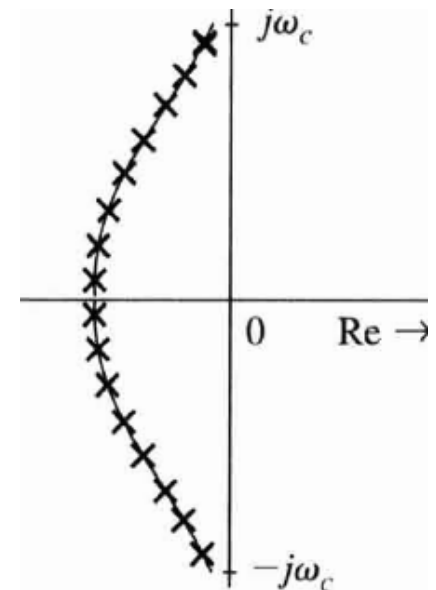
- Observe that by putting ω_c to the numerator we achieve $H(0) = 1$.
- If the distance from the pole to a point $j\omega$ is d then $|H(j\omega)| = \frac{\omega_c}{d}$.



Lowpass filters. Wall of poles – Butterworth filters

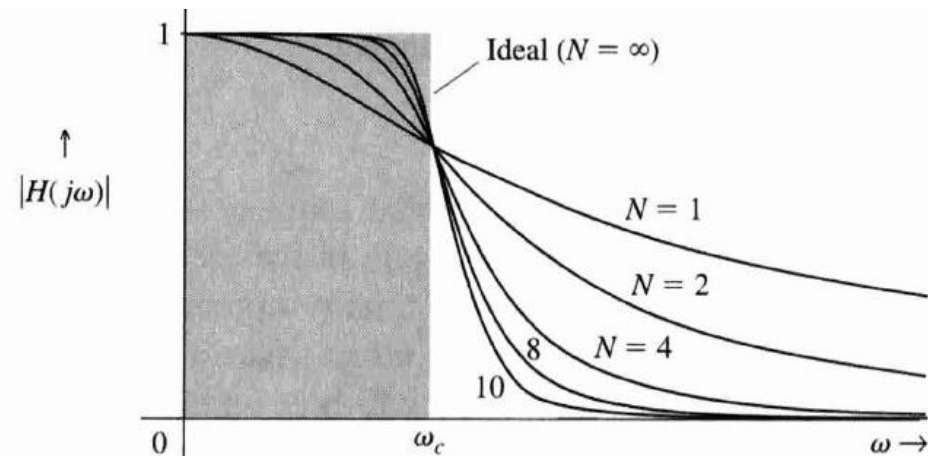
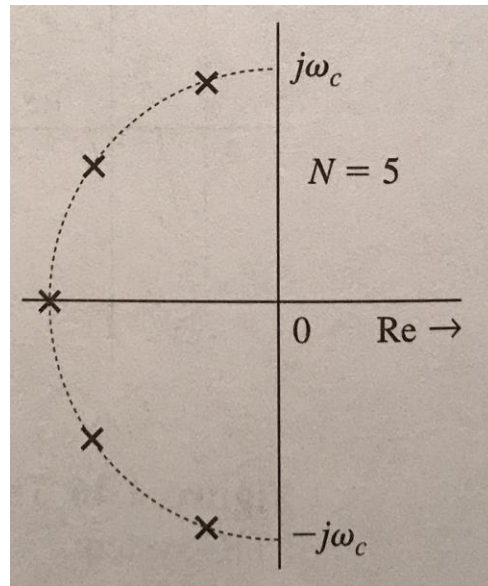
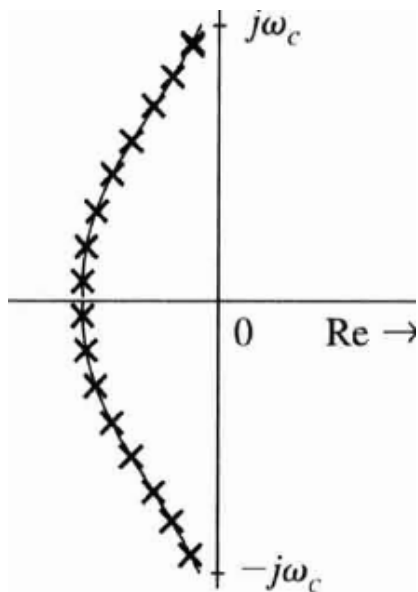
- An ideal lowpass filter has a constant gain of 1 up to a desired frequency ω_c and then the gain drops to 0.
- Therefore, for an ideal lowpass filter an enhanced gain is required within the frequency range 0 to ω_c . This implies that a pole must be placed opposite every single frequency within the range 0 to ω_c .
- We require ideally a continuous “wall of poles” facing the imaginary axis opposite the range 0 to ω_c , and consequently, their complex conjugates facing the imaginary axis opposite the range 0 to $-\omega_c$.
 - At this stage we are not interested in investigating the optimal shape of this wall of poles.
 - We can prove that for a maximally flat response within the range 0 to ω_c , the wall is a semicircle.
 - A maximally flat amplitude response implies:

$$\left. \frac{d^i |H(\omega)|}{d\omega^i} \right|_{\omega=0} = 0, i = 0, \dots, 2N - 1$$



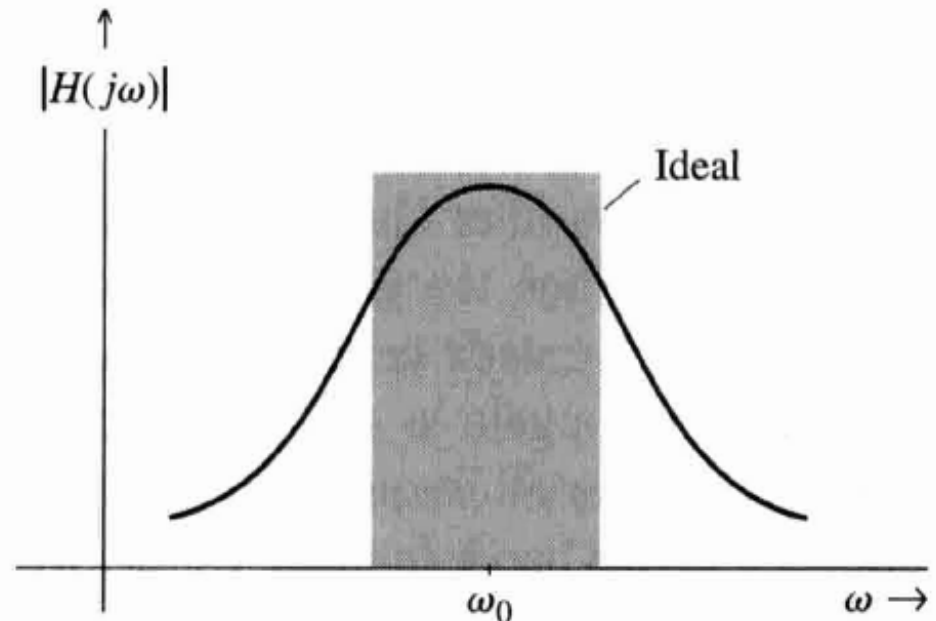
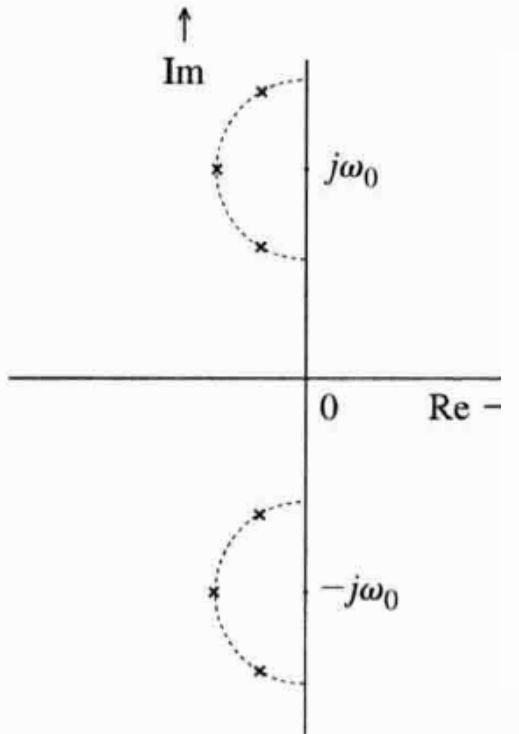
Lowpass filters. Wall of poles – Butterworth filters.

- We can prove that for a maximally flat response within the range 0 to ω_c , the wall is a semicircle with infinite number of poles.
- In practice we use N poles and we end up with a filter with non-ideal characteristics.
- Observe the response as a function of N .
- This family of filters are called Butterworth filters.
- There are families of filters with different characteristics (Chebyshev etc.)



Bandpass filters

- An ideal bandpass filter has a constant gain of 1 placed symmetrically around a desired frequency ω_0 ; otherwise the gain drops to 0.
- Therefore, we require ideally a continuous wall of poles facing the imaginary axis opposite ω_0 , and consequently, their complex conjugates facing the imaginary axis opposite $-\omega_0$.

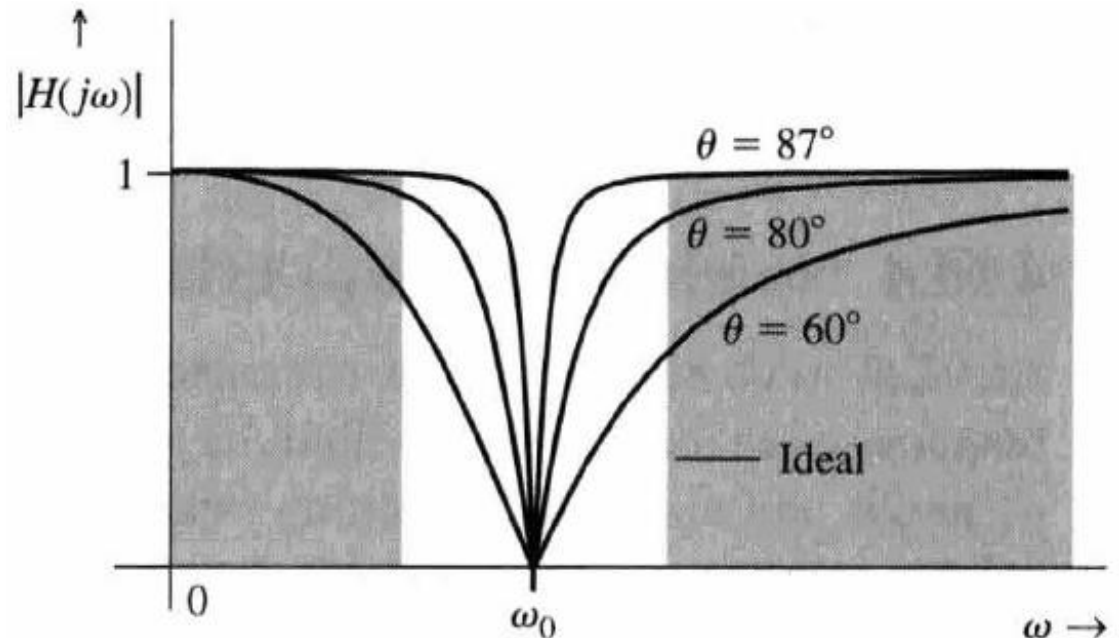
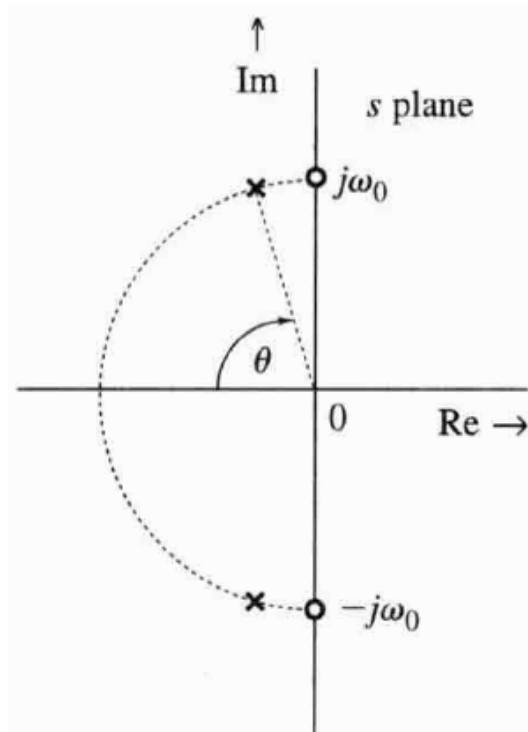


Bandstop (Notch) filters

- An ideal bandstop (notch) filter has 0 amplitude response placed symmetrically around a desired frequency ω_0 ; otherwise the gain is 1.
- Realization in theory requires infinite number of zeros and poles.
- Let us consider a second order notch filter with zero gain at ω_0 .
 - We must have zeros at $\pm j\omega_0$.
 - For $\lim_{\omega \rightarrow \infty} |H(j\omega)| = 1$ the number of poles must be equal to the number of zeros. (For $\omega \rightarrow \infty$ the distance of all poles and zeros from ω is basically the same.)
 - Based on the above two points, we must have two poles.
 - In order to have $|H(0)| = 1$ each pole must pair up with a zero and their distances from the origin must be the same.
 - This requirement can be satisfied if we place the two conjugate poles along a semicircle of radius ω_0 that lies within the left half plane.

Bandstop (Notch) filters cont.

- Based on the previous statements, the pole-zero configuration and the amplitude response of a bandstop filter are shown in the two figures below.
- Observe the behaviour of the amplitude response as a function of θ , the angle that the pole vector forms with the negative real axis.



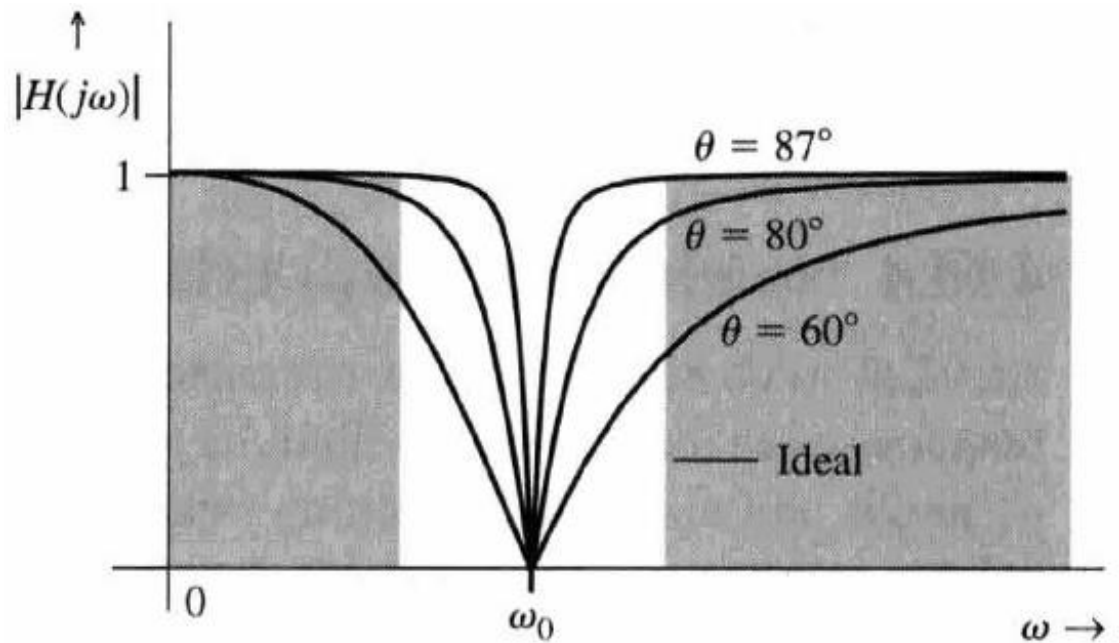
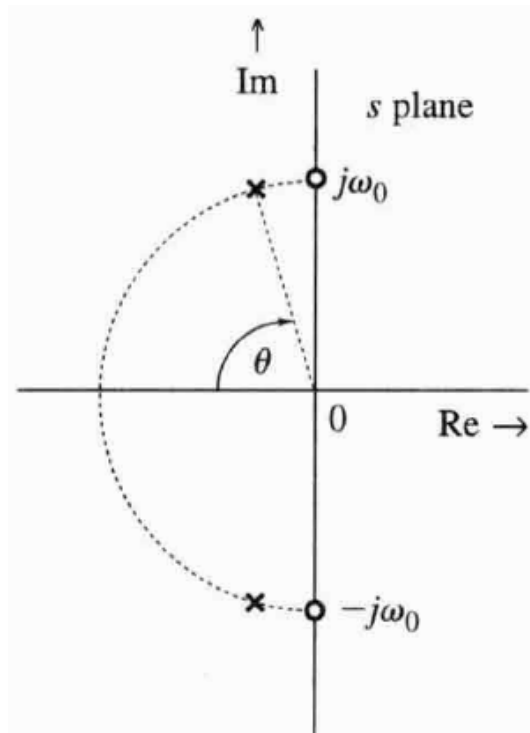
Notch filter example

- Design a second-order notch filter to suppress 60Hz hum in a radio receiver.
- Make $\omega_0 = 120\pi$. Place zeros are at $s = \pm j\omega_0$, and poles at $-\omega_0 \cos\theta \pm j\omega_0 \sin\theta$. We obtain:

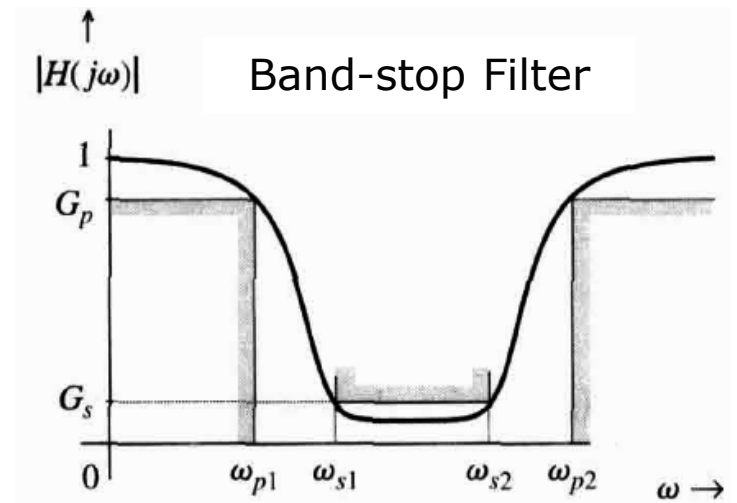
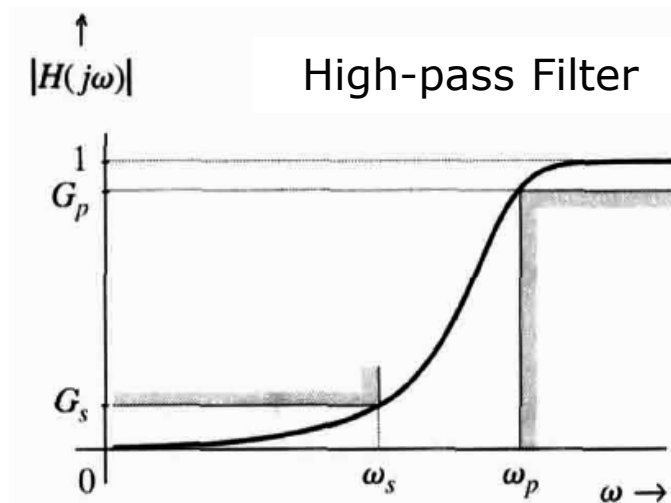
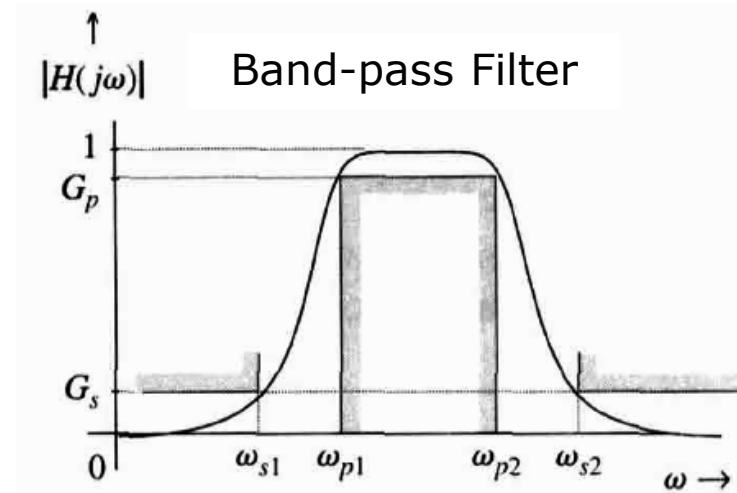
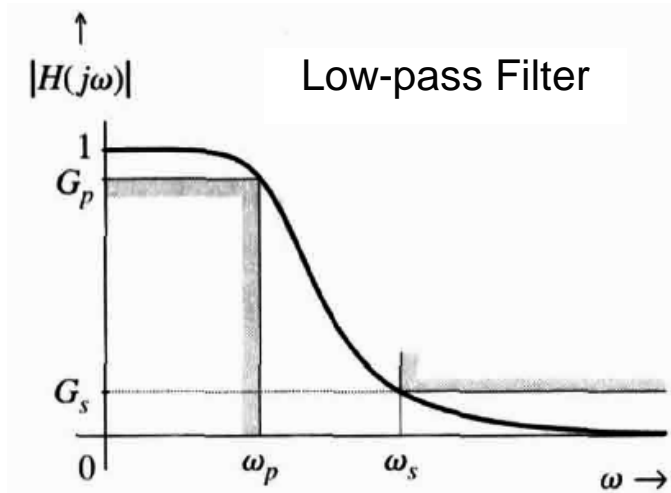
$$\begin{aligned}
 H(s) &= \frac{(s - j\omega_0)(s + j\omega_0)}{(s + \omega_0 \cos\theta + j\omega_0 \sin\theta)(s + \omega_0 \cos\theta - j\omega_0 \sin\theta)} \\
 &= \frac{s^2 + \omega_0^2}{s^2 + (2\omega_0 \cos\theta)s + \omega_0^2} = \frac{s^2 + 142122.3}{s^2 + (753.98 \cos\theta)s + 142122.3} \\
 |H(j\omega)| &= \frac{-\omega^2 + 142122.3}{\sqrt{(-\omega^2 + 142122.3)^2 + (753.98\omega \cos\theta)^2}}
 \end{aligned}$$

Notch filter example cont.

- The figures below depict the location of poles and zeros within the plane and the amplitude response.



Practical filter specification



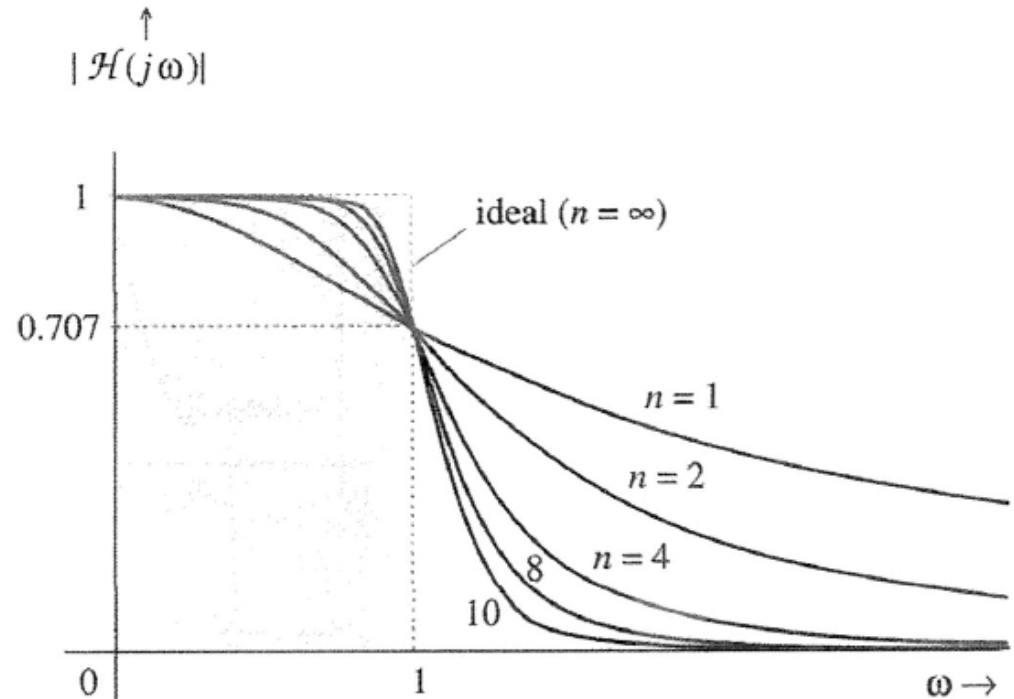
Butterworth filters again

- Let us consider a normalised low-pass filter (i.e., one that has a cut-off frequency at 1) with an amplitude characteristic given by the equation:

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}$$

- As $n \rightarrow \infty$, this gives a ideal LPF response:

- $|H(j\omega)| = 1$ if $\omega \leq 1$
- $|H(j\omega)| = 0$ if $\omega > 1$



Butterworth filters cont.

- In the previous amplitude response we replace $s = j\omega$ and we obtain:

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}} \Rightarrow H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}$$

- The poles of $H(s)H(-s)$ are given by $1 + \left(\frac{s}{j}\right)^{2n} = 0 \Rightarrow \left(\frac{s}{j}\right)^{2n} = -1$.

- We know that $-1 = e^{j\pi(2k-1)}$ and $j = e^{j\frac{\pi}{2}}$.

- $\left(\frac{s}{j}\right)^{2n} = -1 \Rightarrow s^{2n} = j^{2n} \cdot (-1) = e^{(j\frac{\pi}{2})2n} \cdot e^{j\pi(2k-1)} = e^{j\pi n} \cdot e^{j\pi(2k-1)}$

$$\Rightarrow s^{2n} = e^{j\pi(2k-1+n)} \Rightarrow s = e^{\frac{j\pi(2k-1+n)}{2n}}, k \text{ integer.}$$

- Therefore, the poles of $H(s)H(-s)$ lie along the unit circle (a circle around the origin with radius equal to 1). There are $2n$ distinct poles given by:

$$s_k = e^{\frac{j\pi(2k-1+n)}{2n}}, k = 1, 2, \dots, 2n$$

Butterworth filters cont.

- We are only interested in $H(s)$, not $H(-s)$. Therefore, we choose the poles of the low-pass filter to be those lying on the left half plane only. These poles are:

$$s_k = e^{\frac{j\pi(2k-1+n)}{2n}} = \cos\frac{\pi}{2n}(2k-1+n) + j\sin\frac{\pi}{2n}(2k-1+n), k = 1, 2, \dots, n$$

- The transfer function of the filter is:

$$H(s) = \frac{1}{(s - s_1)(s - s_2) \dots (s - s_N)}$$

- This is a class of filters known as Butterworth filters.

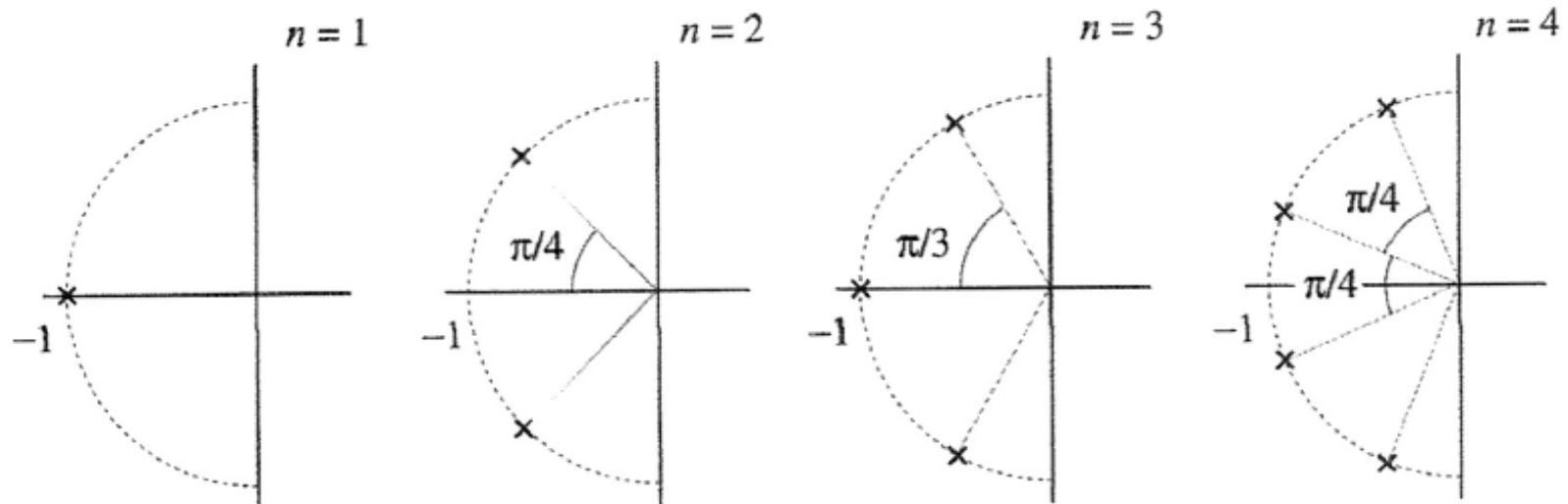
Butterworth filters cont.

- To resume, Butterworth filters are a family of filters with poles distributed evenly around the left half of the unit circle. The poles are given by:

$$s_k = e^{\frac{j\pi(2k+n-1)}{2n}}, k = 1, 2, \dots, n$$

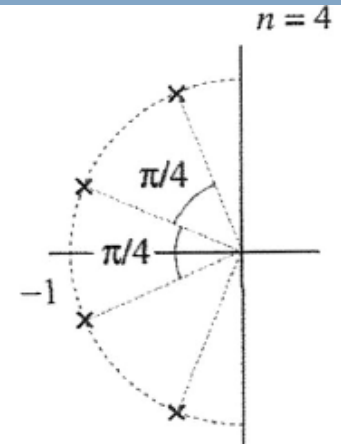
- We assume $\omega_c = 1$.
- Here are the pole locations for Butterworth filters for orders $n = 1$ to 4.

$$H(s) = \frac{1}{(s - s_1)(s - s_2) \dots (s - s_N)}$$



Butterworth filters cont.

- Consider a fourth-order Butterworth filter (i.e., $n = 4$).
- The poles are at angles $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$.
- Therefore, the pole locations are:
 $-0.3827 \pm j0.9239, -0.9239 \pm j0.3827$.



$$\text{Therefore, } H(s) = \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)} = \frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$$

Coefficients of Butterworth polynomial: $B_n(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + 1$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.41421356								
3	2.00000000	2.00000000							
4	2.61312593	3.41421356	2.61312593						
5	3.23606798	5.23606798	5.23606798	3.23606798					
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331				
7	4.49395921	10.09783468	14.59179389	14.59179389	10.09783468	4.49395921			
8	5.12583090	13.13707118	21.84615097	25.68835593	21.84615097	13.13707118	5.12583090		
9	5.75877048	16.58171874	31.16343748	41.98638573	41.98638573	31.16343748	16.58171874	5.75877048	
10	6.39245322	20.43172909	42.80206107	64.88239627	74.23342926	64.88239627	42.80206107	20.43172909	6.39245322

Design of Butterworth filters with Sallen-Key

- The transfer function of the Sallen-Key filter on the right is:

$$H(s) = \frac{1}{1 + C_2(R_1 + R_2)s + C_1C_2R_1R_2s^2}$$

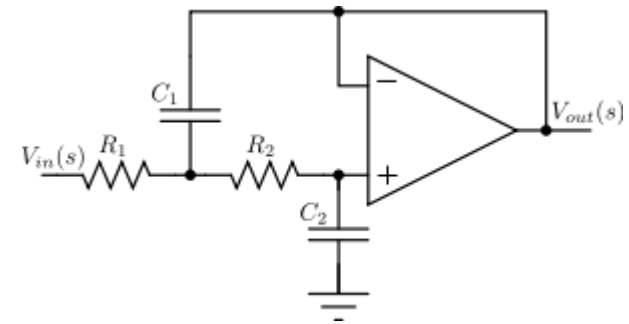
- Assuming that $\omega_c = 1$ and n even we choose:

- $C_1C_2R_1R_2 = 1$
 - $C_2(R_1 + R_2) = -2\cos\left(\frac{2k+n-1}{2n}\pi\right)$

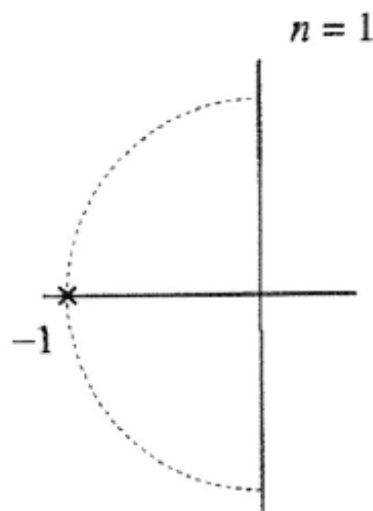
- This guarantees that $H(s)$ has two poles at

$$\cos\left(\frac{2k+n-1}{2n}\pi\right) \pm j \sin\left(\frac{2k+n-1}{2n}\pi\right).$$

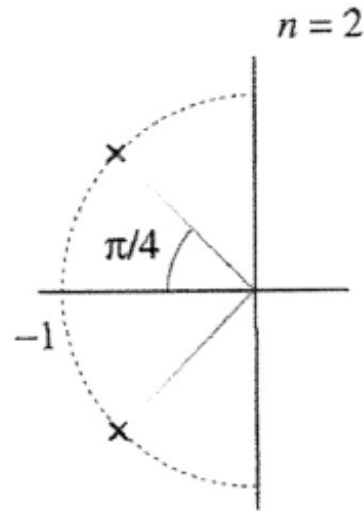
- Cascade $n/2$ such filters.
- When n is odd the remaining real pole can be implemented with an RC circuit.



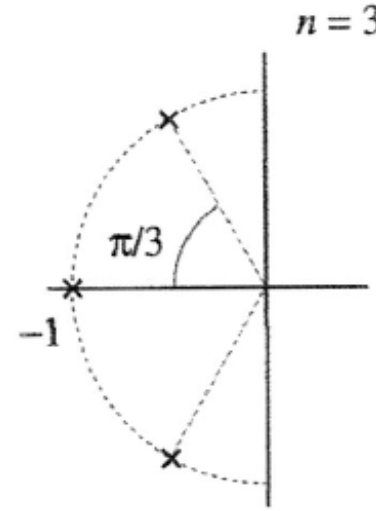
Design of Butterworth filters with Sallen-Key



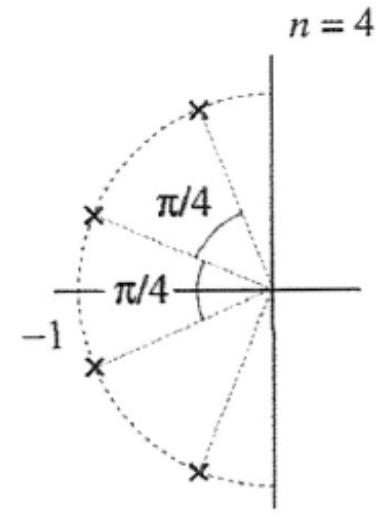
One RC circuit



One Sallen-Key with $k = 1$ and $n = 2$



One Sallen-key with $k = 1$, $n = 3$ followed by one RC circuit



A cascade of two Sallen-key with $n = 4$ and $k = 1, 2$.

Design of Butterworth filters with Sallen-Key

- So far we have considered only normalized Butterworth filters with $3dB$ bandwidth and $\omega_c = 1$.
- We can design filters for any other cut-off frequency by substituting s by s/ω_c .
- For example, the transfer function for a second-order Butterworth filter for $\omega_c = 100$ is given by:

$$H(s) = \frac{1}{\left(\frac{s}{100}\right)^2 + \sqrt{2}\left(\frac{s}{100}\right) + 1} = \frac{1}{s^2 + 100\sqrt{2}s + 10^4}$$

Relating this lecture to other courses

- You will learn about poles and zeros in your 2nd year control course. The emphasis here is to provide you with intuitive understanding of their effects on frequency response.
- You will probably do Butterworth filters in your 2nd year analogue circuits course.
- Some of you will be implementing the notch filter in your 3rd year on a real-time digital signal processor (depending on options you take), and others will learn more about filter design in your 3rd and 4th year.