Imperial College London

Signals and Systems

Lecture 5

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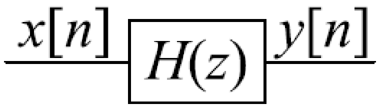
Difference Equations

Most useful LTI systems can be described by a difference equation:



$$y[n] = \sum_{r=0}^{M} b[r]x[n-r] - \sum_{r=1}^{N} a[r]y[n-r]$$

$$\Leftrightarrow \sum_{r=0}^{N} a[r]y[n-r] = \sum_{r=0}^{M} b[r]x[n-r]$$ with $a[0] = 1$

$$\Leftrightarrow a[n] * y[n] = b[n] * x[n]$$

$$\Leftrightarrow Y(z) = \frac{B(z)}{A(z)} X(z)$$

$$\Leftrightarrow Y(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})} X(e^{j\omega})$$

(1) Always causal.
(2) Order of system is $\max(M, N)$, the highest r with $a[r] \neq 0$ or $b[r] \neq 0$.
(3) We assume that $a[0] = 1$; if not, divide $A(z)$ and $B(z)$ by $a[0]$.
(4) Filter is BIBO stable iff roots of $A(z)$ all lie within the unit circle.

FIR Filters

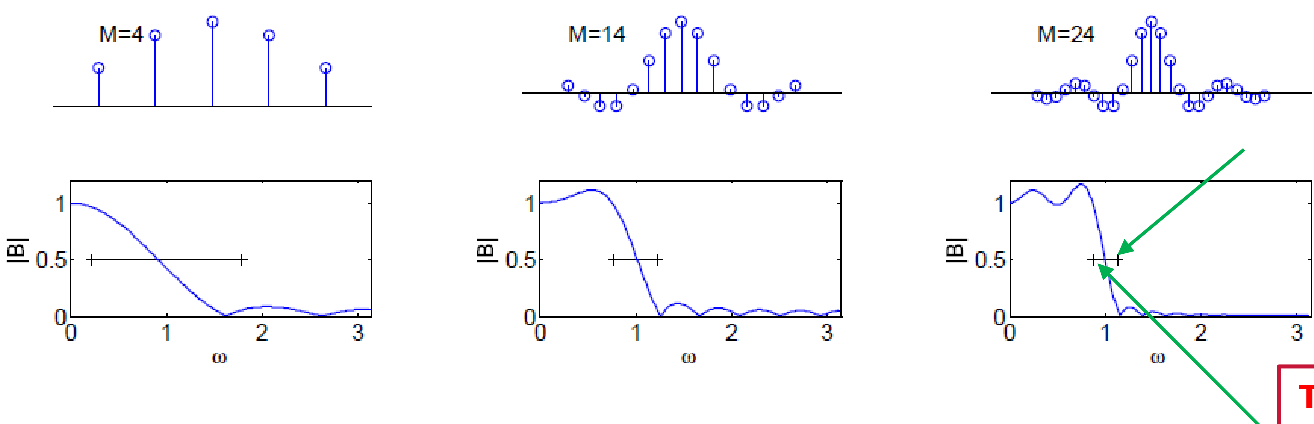
$A(z) = 1$: Finite Impulse Response (FIR) filter: $Y(z) = B(z)X(z)$.
Impulse response is $b[n]$ and is of length $M + 1$.

Frequency response is $B(e^{j\omega})$ and is the DTFT of $b[n]$.
Comprises M complex sinusoids + const:

$$b[0] + b[1]e^{-j\omega} + \cdots + b[M]e^{-jM\omega}$$

Small $M \Rightarrow$ response contains only low "quefrencies"

Symmetrical $b[n] \Rightarrow H(e^{j\omega})e^{\frac{jM\omega}{2}}$ consists of $\frac{M}{2}$ cosine waves [+ const]



The two marked points around 1 are $1 \pm \frac{2\pi}{M}$.

Rule of thumb: Fastest possible transition $\Delta\omega \geq \frac{2\pi}{M}$ (marked line)

FIR Symmetries

$B(e^{j\omega})$ is determined by the zeros of $z^M B(z) = \sum_{r=0}^{M} b[M-r]z^r$

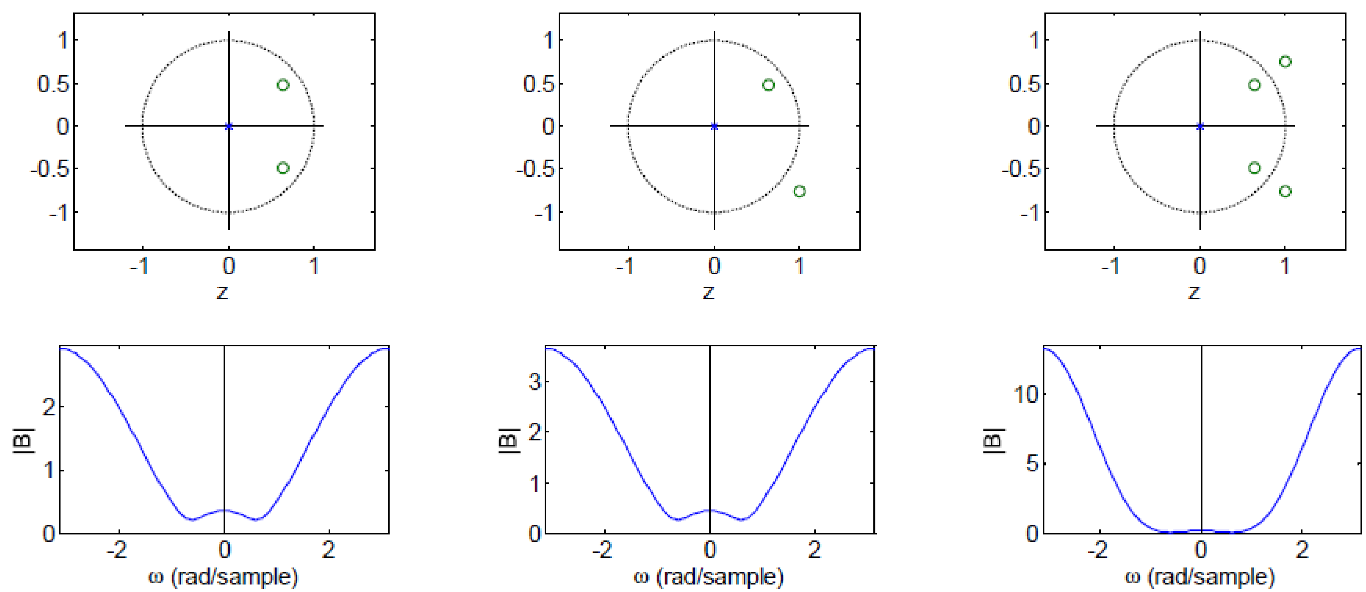
Sup

Real $b[n]$ $\Rightarrow$ conjugate zero pairs: $z \Rightarrow z^*$

Symmetric: $b[n] = b[M-n]$ $\Rightarrow$ reciprocal zero pairs: $z \Rightarrow z^{-1}$

Real + Symmetric $b[n]$ $\Rightarrow$ conjugate+reciprocal groups of four or else pairs on the real axis

Real:	Symmetric:	Real + Symmetric:
$[1,\ -1.28,\ 0.64]$	$[1,\ -1.64+0.27j,\ 1]$	$[1,\ -3.28,\ 4.7625,\ -3.28,\ 1]$



FIR Symmetries

In all of the proofs below, we assume that $z = z_0$ is a root of $B(z)$ so that $B(z_0) = \sum_{r=0}^{M} b[r] z_0^{-r} = 0$ and then we prove that this implies that other values of z also satisfy $B(z) = 0$.

(1) Real $b[n]$

$$\begin{aligned} B(z_0^*) &= \textstyle\sum_{r=0}^{M} b[r] \left(z_0^*\right)^{-r} \\ &= \textstyle\sum_{r=0}^{M} b^*[r] \left(z_0^*\right)^{-r} && \text{since } b[r] \text{ is real} \\ &= \left(\textstyle\sum_{r=0}^{M} b[r] z_0^{-r}\right)^* && \text{take complex conjugate} \\ &= 0^* = 0 && \text{since } B(z_0) = 0 \end{aligned}$$

(2) Symmetric: $b[n] = b[M-n]$

$$\begin{aligned} B(z_0^{-1}) &= \textstyle\sum_{r=0}^{M} b[r] z_0^{r} \\ &= \textstyle\sum_{n=0}^{M} b[M-n] z_0^{M-n} && \text{substitute } r = M-n \\ &= z_0^M \textstyle\sum_{n=0}^{M} b[M-n] z_0^{-n} && \text{take out } z_0^M \text{ factor} \\ &= z_0^M \textstyle\sum_{n=0}^{M} b[n] z_0^{-n} && \text{since } b[M-n] = b[n] \\ &= z_0^M \times 0 = 0 && \text{since } B(z_0) = 0 \end{aligned}$$

IIR Frequency Response

Factorize $H(z) = \frac{B(z)}{A(z)} = \frac{b[0]\prod_{i=1}^{M}\left(1-q_i z^{-1}\right)}{\prod_{i=1}^{N}\left(1-p_i z^{-1}\right)}$

Roots of $A(z)$ and $B(z)$ are the "poles" $\{p_i\}$ and "zeros" $\{q_i\}$ of $H(z)$
Also an additional $N-M$ zeros at the origin (affect phase only)

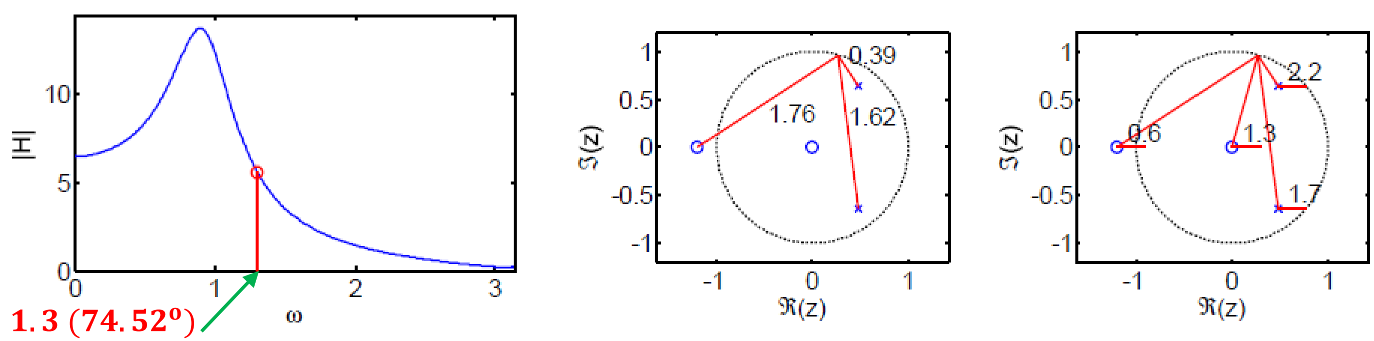
$$\left|H(e^{j\omega})\right| = \frac{|b[0]|\left|z^{-M}\right|\prod_{i=1}^{M}|z-q_i|}{\left|z^{-N}\right|\prod_{i=1}^{N}|z-p_i|} \text{ for } z = e^{j\omega}$$

Example:

$$H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}} = \frac{2\left(1+1.2z^{-1}\right)}{(1-(0.48-0.64j)z^{-1})(1-(0.48+0.64j)z^{-1})}$$

At $\omega = 1.3$: $\left|H(e^{j\omega})\right| = \frac{2\times 1.76}{1.62\times 0.39} = 5.6$

$$\angle H(e^{j\omega}) = (0.6+1.3)-(1.7+2.2) = -2$$



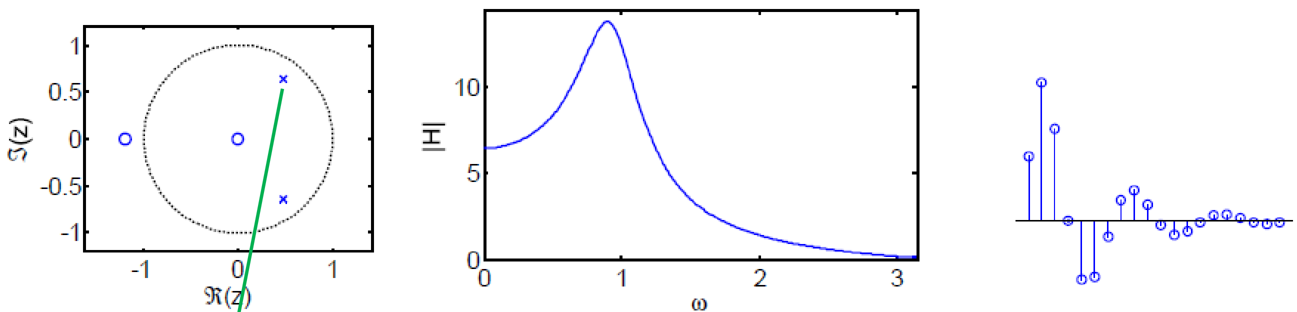
$N-M = 2-1 = 1$
(one zero at the origin)

Negating z

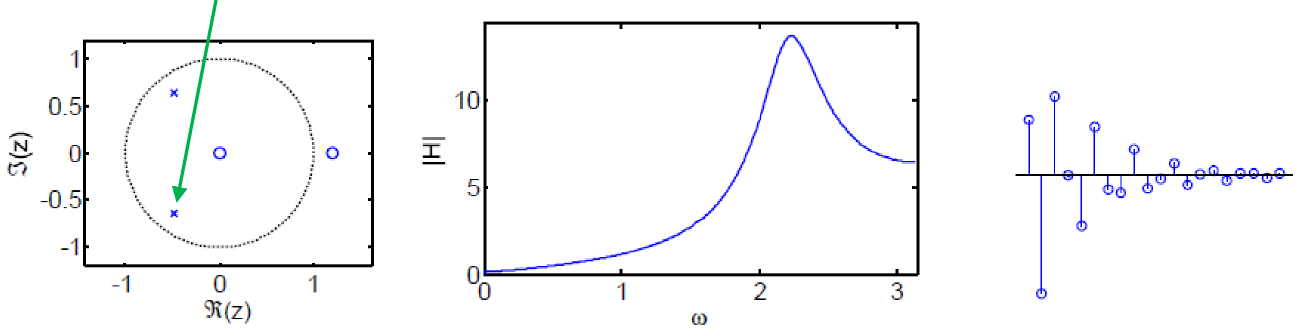
Given a filter $H(z)$ we can form a new one $H_R(z) = H(-z)$

Negate all odd powers of z, i.e. negate alternate $a[n]$ and $b[n]$

Example: $H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}}$



Negate z: $H_R(z) = \frac{2-2.4z^{-1}}{1+0.96z^{-1}+0.64z^{-2}}$ Negate odd coefficients



Pole and zero positions are negated, response is flipped and conjugated.

Negating z cont.

Suppose that $H_R(z) = H(-z)$. Then $H_R(z)$ has the following two properties:

Pole and zero positions are negated

If z_0 is a zero of $H(z)$, then $H_R(-z_0) = H(z_0) = 0$ so $-z_0$ is a zero of $H_R(z)$. A similar argumnet applies to poles.

The frequency response is flipped and conjugated

The frequency response is given by $H_R(e^{j\omega}) = H(-e^{j\omega}) = H(e^{-j\pi} \times e^{j\omega}) = H(e^{j(\omega-\pi)})$. This corresponds to shifting the frequency reponse by $\pi\,\mathrm{rad/samp}$ (or, equivalently by $-\pi\,\mathrm{rad/samp}$).

If it is true that all the coefficients in $a[n]$ and $b[n]$ are real-valued (normally the case), then the response of $H(z)$ has conjugate symmetry, i.e. $H(e^{-j\omega}) = H^*(e^{j\omega})$. In this case we can write $H_R(e^{j\omega}) = H(e^{j(\omega-\pi)}) = H^*(e^{j(\pi-\omega)})$. This corresponds to a frequency response that has been reflected around $\omega = \frac{\pi}{2}$ (a.k.a. "flipped") and then conjugated. $\omega = \frac{\pi}{2} + x, H\left(e^{j\left(\frac{\pi}{2}+x-\pi\right)}\right) = H\left(e^{j\left(x-\frac{\pi}{2}\right)}\right)$

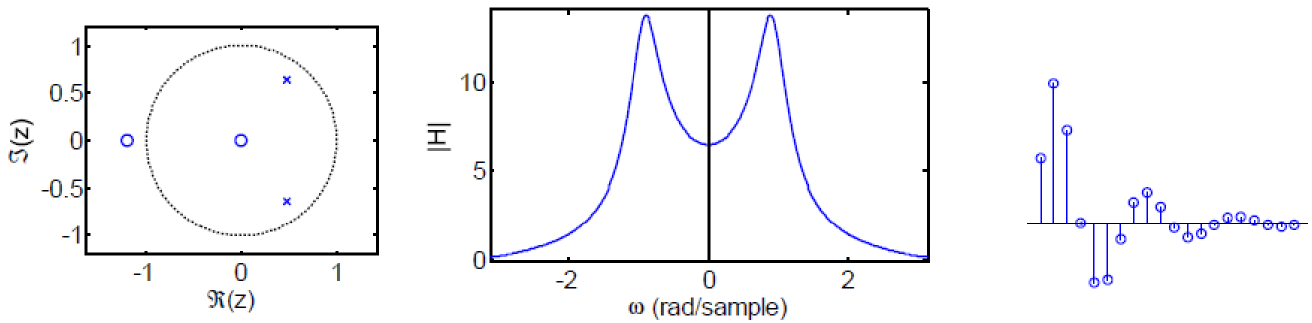
So, the transformation of the frequency can be viewed in one of two ways: (a) it has been shifted by $\pm\pi\,\mathrm{rad/samp}$ or (b) it has been flipped around $\omega = \frac{\pi}{2}$ and then conjugated. The first interpretation is always true (even for filters with complex-valued coefficients) while the second interpretation is more intuitive but is only true if the filter coefficients are real-valued.

Cubing z

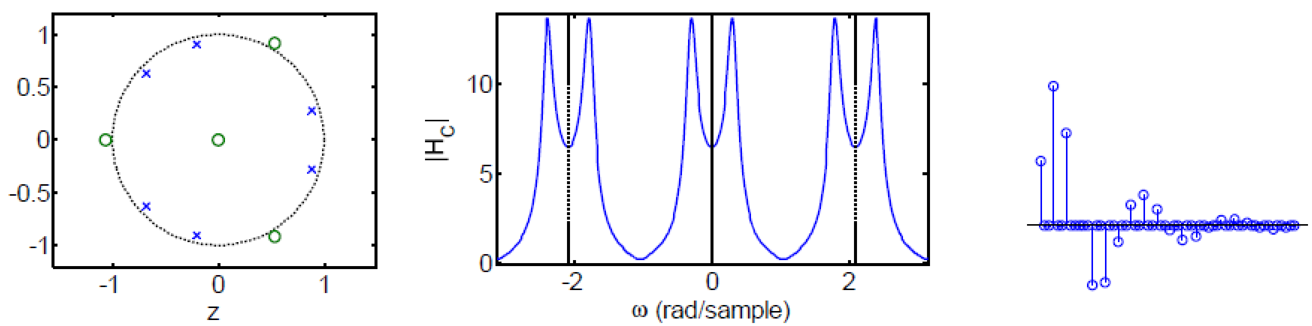
Given a filter $H(z)$ we can form a new one $H_C(z) = H(z^3)$

Insert two zeros between each $a[n]$ and $b[n]$ term

Example: $H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}}$



Cube z: $H_C(z) = \frac{2+2.4z^{-3}}{1-0.96z^{-3}+0.64z^{-6}}$ Insert 2 zeros between coefs



Pole and zero positions are replicated, magnitude response replicated.

Cubing z

Suppose that $H_C(z) = H(z^3)$. Then $H_C(z)$ has the following two properties:

Pole and zero positions are replicated three times

If z_0 is a zero of $H(z)$, then $H_C(\sqrt[3]{z_0}) = H(z_0) = 0$ so any cube root of z_0 is a zero of $H_C(z)$. A similar argument applies to poles. Any z_0 has three cube roots in the complex plane whose magnitudes all have the same value of $\sqrt[3]{|z_0|}$ and whose arguments are $\angle z_0 + \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$.

The frequency response is replicated three times

The frequency response is given by $H_C(e^{j\omega}) = H(e^{j3\omega})$. This corresponds to shrinking the response horizontally by a factor of 3. Also $H_C\left(e^{j\left(\omega \pm \frac{2\pi}{3}\right)}\right) = H\left(e^{j3\left(\omega \pm \frac{2\pi}{3}\right)}\right) = H\left(e^{j3\omega \pm 2\pi}\right) = H_C\left(e^{j\omega}\right)$ meaning that there are three replications of the frequency response spaced $\frac{2\pi}{3}$ apart. Note that if you only look at the positive frequencies, there are three replications of the positive half of the reponse but alternate copies are flipped and conjugated (assuming the coefficients $a[n]$ and $b[n]$ are real-valued).

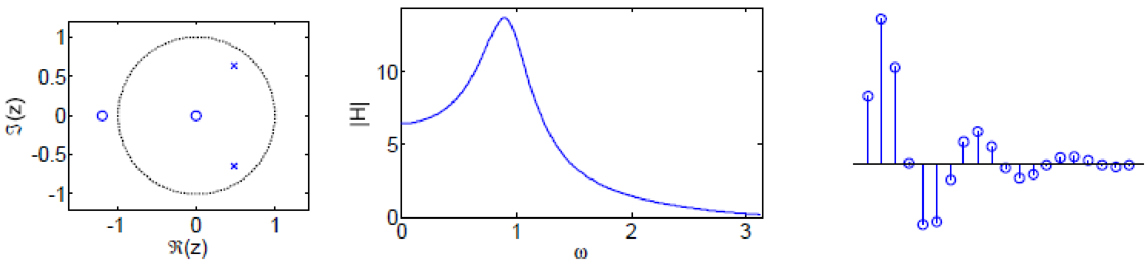
All of this carries over to raising z to any positive integer power; the number of replications is equal to the power concerned.

Scaling z

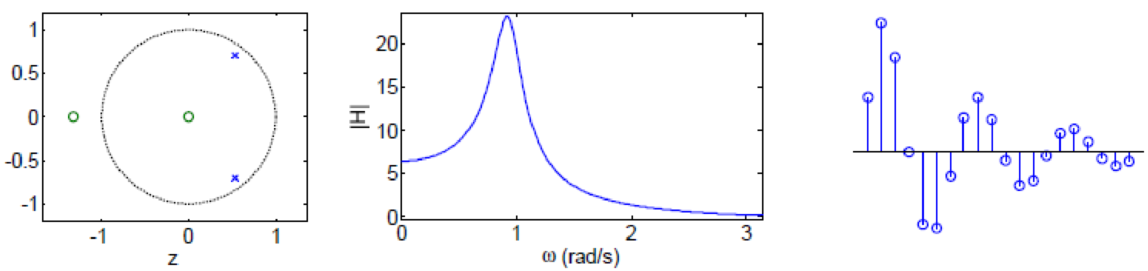
Given a filter $H(z)$ we can form a new one $H_S(z) = H(\frac{z}{\alpha})$

Multiply $a[n]$ and $b[n]$ by α^n

Example: $H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}}$



Scale z: $H_S(z) = H(\frac{z}{1.1}) = \frac{2+2.64z^{-1}}{1-1.056z^{-1}+0.7744z^{-2}}$



Pole and zero positions are multiplied by α, $\alpha > 1 \Rightarrow$peaks sharpened.

Pole at $z = p$ gives peak bandwidth $\approx 2\,|\log|p|| \approx 2\,(1-|p|)$

For pole near unit circle, decrease bandwidth by $\approx 2\log\alpha$

Scaling z

Suppose that $H_S(z) = H\left(\frac{z}{\alpha}\right)$ where α is a non-zero real number. Then $H_S(z)$ has the following two properties:

Pole and zero positions are multiplied by α

If z_0 is a zero of $H(z)$, then $H_S(\alpha z) = H(z_0) = 0$ so αz_0 is a zero of $H_S(z)$. The argument of the zero is unchanged since $\angle \alpha z_0 = \angle z_0$. The magnitude of the zero is multiplied by α. A similar argument applies to poles. If $\alpha > 1$ then the pole positions will move closer to the unit circle. If α is large enough to make any pole cross the unit circle then the filter $H_S(z)$ will be unstable.

The bandwidth of any peaks in the response are decreased by approximately $2 \log \alpha$

If $H(z)$ has a pole, p, that is near the unit circle, it results in a peak in the magnitude response at $\omega = \angle p$ whose amplitude is proportional to $\frac{1}{1-|p|}$ and whose bandwidth is approximately equal to $-2 \log |p| \approx 2(1 - |p|)$ (which is positive since $|p| < 1$). The corresponding pole in $H_S(z)$ is at αp, so its approximate bandwidth is now $-2 \log |\alpha p| = -2 \log |p| - 2 \log \alpha$. Thus the bandwidth has decreased by about $2 \log \alpha$.

If $\alpha > 1$ then $\log \alpha$ is positive and the peak in $H_S(z)$ will have a higher amplitude and a smaller bandwidth. If $\alpha < 1$, then $\log \alpha$ is negative and the peak will have a lower amplitude and a larger bandwidth.

Low-pass filter

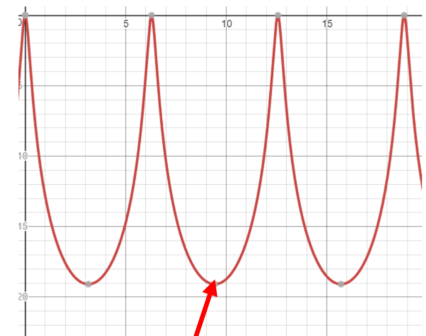
1st order low pass filter: extremely common

$$y[n] = (1-p)x[n] + py[n-1] \Rightarrow H(z) = \frac{1-p}{1-pz^{-1}}$$

Impulse response:

$$h[n] = (1-p)p^n = (1-p)e^{-\frac{n}{\tau}}$$

where $\tau = \frac{1}{-\ln p}$ is the time constant in samples.



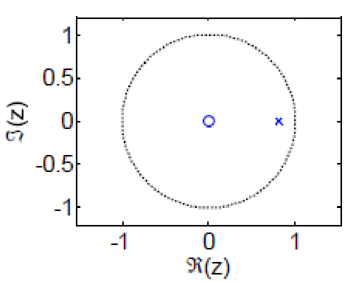
Magnitude response: $\left|H(e^{j\omega})\right| = \frac{1-p}{\sqrt{1-2p\cos\omega+p^2}}$

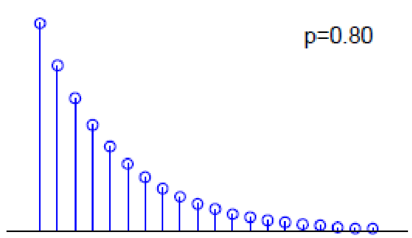
Low-pass filter with DC gain of unity.

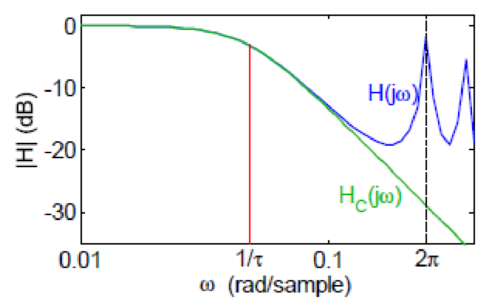
3 dB frequency is $\omega_{3dB} = \cos^{-1}\left(1 - \frac{(1-p)^2}{2p}\right) \approx 2\frac{1-p}{1+p} \approx \frac{1}{\tau}$

Compare continuous time: $H_C(j\omega) = \frac{1}{1+j\omega\tau}$

Indistinguishable for low ω but $H(e^{j\omega})$ is periodic, $H_C(j\omega)$ is not







The red and blue curves are the same but in the blue curve the logarithmic scale is depicted in both axes.

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3 db approximation

To find the 3dB frequency we require $|H(e^{j\omega_3})| = \sqrt{\frac{1}{2}} \Leftrightarrow |H(e^{j\omega_0})|^2 = \frac{1}{2}$.

$$\frac{(1-p)^2}{1-2p\cos\omega_3+p^2} = \frac{1}{2}$$

$$\Rightarrow 2(1-p)^2 = 1 - 2p\cos\omega_3 + p^2$$

$$\Rightarrow 2(1-p)^2 = (1-p)^2 + 2p(1-\cos\omega_3)$$

$$\Rightarrow \cos\omega_3 = 1 - \frac{(1-p)^2}{2p}$$

$$\Rightarrow \omega_3 = \cos^{-1}\left(1 - \frac{(1-p)^2}{2p}\right)$$

$$\boldsymbol{\omega = \cos^{-1}(x) = \sqrt{2-2x}}$$

Expressing $\cos\omega = x$ as a Taylor series gives $x \approx 1 - \frac{\omega^2}{2} \Rightarrow \omega \approx \sqrt{2-2x}$. So replacing x by the expression in parentheses gives $\omega_3 \approx \sqrt{\frac{(1-p)^2}{p}} = \frac{1-p}{\sqrt{p}}$.

Writing $d = 1 - p$ and assuming d is small, we can write $\sqrt{p} = (1-d)^{\frac{1}{2}} \approx 1 - \frac{1}{2}d = \frac{1}{2}(1+p)$. Substituting this into the previous expression gives $\omega_3 \approx 2\frac{1-p}{1+p}$.

Allpass filters

If $H(z) = \frac{B(z)}{A(z)}$ with $b[n] = a^*[M-n]$ then we have an allpass filter:

$$\Rightarrow H(e^{j\omega}) = \frac{\sum_{r=0}^{M} a^*[M-r]e^{-j\omega r}}{\sum_{r=0}^{M} a[r]e^{-j\omega r}} = e^{-j\omega M}\frac{\sum_{s=0}^{M} a^*[s]e^{j\omega s}}{\sum_{r=0}^{M} a[r]e^{-j\omega r}} \qquad [s = M - r]$$

The two sums are complex conjugates $\Rightarrow$ they have the same magnitude

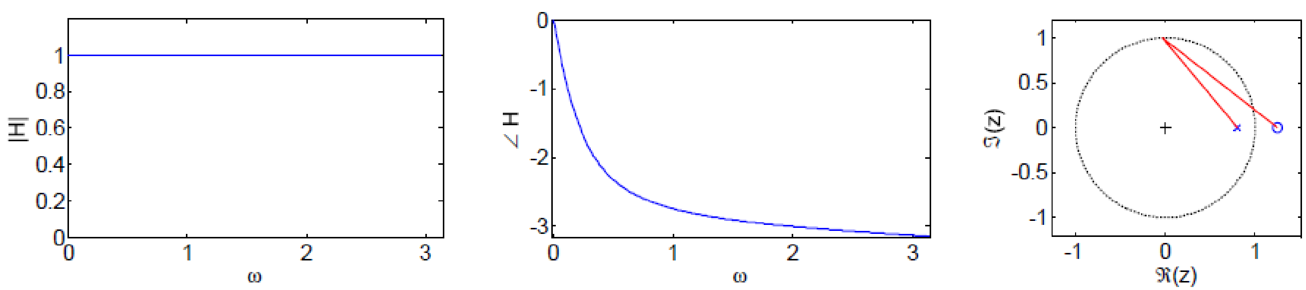
Hence $\left|H(e^{j\omega})\right| = 1 \forall\omega \Leftrightarrow$ "allpass"

However phase is not constant: $\angle H(e^{j\omega}) = -\omega M - 2\angle A(e^{j\omega})$

1st order allpass: $H(z) = \frac{-p+z^{-1}}{1-pz^{-1}} = -p\frac{1-p^{-1}z^{-1}}{1-pz^{-1}}$

Pole at p and zero at p^{-1}: "reflected in unit circle"

Constant distance ratio: $\left|e^{j\omega} - p\right| = |p|\left|e^{j\omega} - \frac{1}{p}\right| \forall\omega$



In an allpass filter, the zeros are the poles reflected in the unit circle.

Allpass filters properties

An allpass filter is one in which $H(z) = \frac{B(z)}{A(z)}$ with $b[n] = a^*[M - n]$. Of course, if the coefficients $a[n]$ are all real, then the conjugation has no effect and the numerator coefficients are identical to the numerator coefficients but in reverse order.

If $A(z)$ has order M, we can express the relation between $A(z)$ and $B(z)$ algebraically as $B(z) = z^{-M}\bar{A}(z^{-1})$ where the coefficients of $\bar{A}(z)$ are the conjugates of the coefficients of $A(z)$.

If the roots of $A(z)$ are p_i, then we can express $H(z)$ in factorized form as

$$H(z) = \prod_{i=1}^{M} \frac{-p_i^* + z^{-1}}{1 - p_i z^{-1}} = \prod_{i=1}^{M} \frac{1 - p_i^* z}{z - p_i}$$

We can therefore write

$$\begin{aligned} |H(z)|^2 &= \prod_{i=1}^{M} \frac{(1 - p_i^* z)(1 - p_i z^*)}{(z - p_i)(z^* - p_i^*)} = \prod_{i=1}^{M} \frac{1 - p_i z^* - p_i^* z + p_i p_i^* z z^*}{z z^* - p_i z^* - p_i^* z + p_i p_i^*} \\ &= \prod_{i=1}^{M} \left(1 + \frac{1 + p_i p_i^* z z^* - z z^* - p_i p_i^*}{z z^* - p_i z^* - p_i^* z + p_i p_i^*}\right) = \prod_{i=1}^{M} \left(1 + \frac{\left(1 - |z|^2\right)\left(1 - |p_i|^2\right)}{|z - p_i|^2}\right) \end{aligned}$$

If all the $|p_i| < 1$, then each term in the product is $\gtreqless 1$ according to whether $|z| \lesseqgtr 1$.

It follows that, provided $H(z)$ is stable, $|H(z)| \gtreqless 1$ according to whether $|z| \lesseqgtr 1$.

Group delay

Group delay: $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$ = delay of the modulation envelope.

Trick to get at phase: $\ln H(e^{j\omega}) = \ln\left|H(e^{j\omega})\right| + j\angle H(e^{j\omega})$

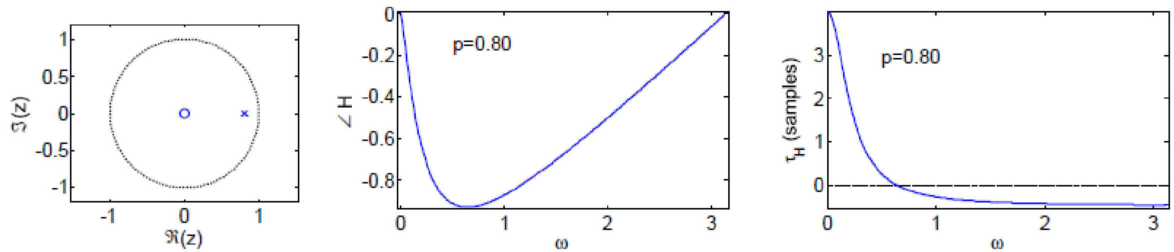
$$\tau_H = \frac{-d\left(\Im\left(\ln H(e^{j\omega})\right)\right)}{d\omega} = \Im\left(\frac{-1}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right) = \Re\left(\frac{-z}{H(z)}\frac{dH}{dz}\right)\Big|_{z=e^{j\omega}}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} h[n]e^{-jn\omega} = \mathscr{F}(h[n]) \qquad [\mathscr{F} = \text{DTFT}]$$

$$\frac{dH(e^{j\omega})}{d\omega} = \sum_{n=0}^{\infty} -jnh[n]e^{-jn\omega} = -j\mathscr{F}(nh[n])$$

$$\tau_H = \Im\left(\frac{-1}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right) = \Im\left(\frac{j\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right) = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$$

Example: $H(z) = \frac{1}{1-pz^{-1}} \Rightarrow \tau_H = -\tau_{[1\ -p]} = -\Re\left(\frac{-pe^{-j\omega}}{1-pe^{-j\omega}}\right)$



Average group delay (over ω) = (# poles – # zeros) within the unit circle

Group delay properties

The group delay of a filter $H(z)$ at a frequency ω gives the time delay (in samples) of the envelope of a modulated sine wave at a frequency ω. It is defined as $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$. For example, $H(z) = z^{-k}$ defines a filter that delays its input by k samples and we can calculate the group delay by evaluating

$$\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = -\frac{d}{d\omega}\left(\angle e^{-jk\omega}\right) = -\frac{d}{d\omega}(-k\omega) = k$$

which tells us that this filter has a constant group delay of k samples that is independent of ω.

The average value of τ_H equals the total change in $-\angle H(e^{j\omega})$ as ω goes from $-\pi$ to $+\pi$ divided by 2π. If you imagine an elastic string connecting a pole or zero to the point $z = e^{j\omega}$, you can see that as ω goes from $-\pi$ to $+\pi$ the string will wind once around the pole or zero if it is inside the unit circle but not if it is outside. Thus, the total change in $\angle H(e^{j\omega})$ is equal to 2π times the the difference between the number of poles and the number of zeros inside the unit circle. A zero that is exactly on the unit circle counts $\frac{1}{2}$ since there is a sudden discontinuity of π in $\angle H(e^{j\omega})$ as ω passes through the zero position.

When you multiply or divide complex numbers, their phases add or subtract, so it follows that when you multiply or divide transfer functions their group delays will add or subtract. Thus, for example, the group delay of an IIR filter, $H(z) = \frac{B(z)}{A(z)}$, is given by $\tau_H = \tau_B - \tau_A$. This means too that we can determine the group delay of a factorized transfer function by summing the group delays of the individual factors.

Group delay from h[n] or H(z)

The slide shows how to determine the group delay, τ_H, from either the impulse response, $h[n]$, or the transfer function, $H(z)$. We start by using a trick that is very common: if you want to get at the magnitude and phase of a complex number separately, you can do so by taking its natural log: $\ln\left(re^{j\theta}\right) = \ln|r| + j\theta$ or, in general, $\ln H = \ln|H| + j\angle H$. By rearranging this equation, we get $\angle H = \Im\left(\ln H\right)$ where $\Im(\,)$ denotes taking the imaginary part of a complex number. Using this, we can write

$$\tau_H = \frac{-d\left(\Im\left(\ln H(e^{j\omega})\right)\right)}{d\omega} = \Im\left(\frac{-d\left(\ln H(e^{j\omega})\right)}{d\omega}\right) = \Im\left(\frac{-1}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right). \quad (1)$$

By going back to the definition of the DTFT, we find that $H(e^{j\omega}) = \mathscr{F}(h[n])$ and $\frac{dH(e^{j\omega})}{d\omega} = -j\mathscr{F}(nh[n])$ where $\mathscr{F}(\,)$ denotes the DTFT. Substituting these expressions into the above equation gives us a formula for τ_H in terms of the impulse response $h[n]$.

$$\tau_H = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right) \quad (2)$$

In order to express τ_H in terms of z, we first note that if $z = e^{j\omega}$ then $\frac{dz}{d\omega} = jz$. By substituting $z = e^{j\omega}$ into equation (1), we get

$$\tau_H = \Im\left(\frac{-1}{H(z)}\frac{dH(z)}{d\omega}\right) = \Im\left(\frac{-1}{H(z)}\frac{dH(z)}{dz}\frac{dz}{d\omega}\right) = \Im\left(\frac{-jz}{H(z)}\frac{dH(z)}{dz}\right) = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\bigg|_{z=e^{j\omega}}.$$

Group delay example

As an example, suppose we want to determine the group delay of : $H(z) = \frac{1}{1-pz^{-1}}$. As noted above, if $H(z) = \frac{B(z)}{A(z)}$, then $\tau_H = \tau_B - \tau_A$. In this case $\tau_B = 0$ so $\tau_H = -\tau_{[1\ -p]}$.

Using equation (2) gives $\tau_H = -\Re\left(\frac{\mathscr{F}([0\ -p])}{\mathscr{F}([1\ -p])}\right)$ since $nh[n] = [0\ 1] \times [1\ \ -p]$.

Applying the definition of the DTFT, we get

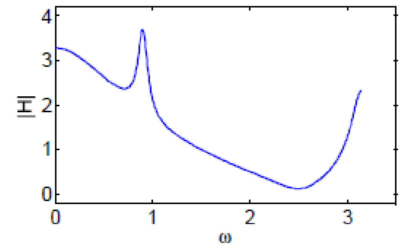
$$\tau_H = -\Re\left(\frac{-pe^{-j\omega}}{1-pe^{-j\omega}}\right) = \Re\left(\frac{p}{e^{j\omega}-p}\right) = \frac{\Re\left(p\left(e^{-j\omega}-p\right)\right)}{\left(e^{j\omega}-p\right)\left(e^{-j\omega}-p\right)} = \frac{p\cos\omega - p^2}{1-2p\cos\omega + p^2}$$

As demonstrated above, the average value of τ_H is zero for this filter because there is one pole and one zero inside the unit circle.

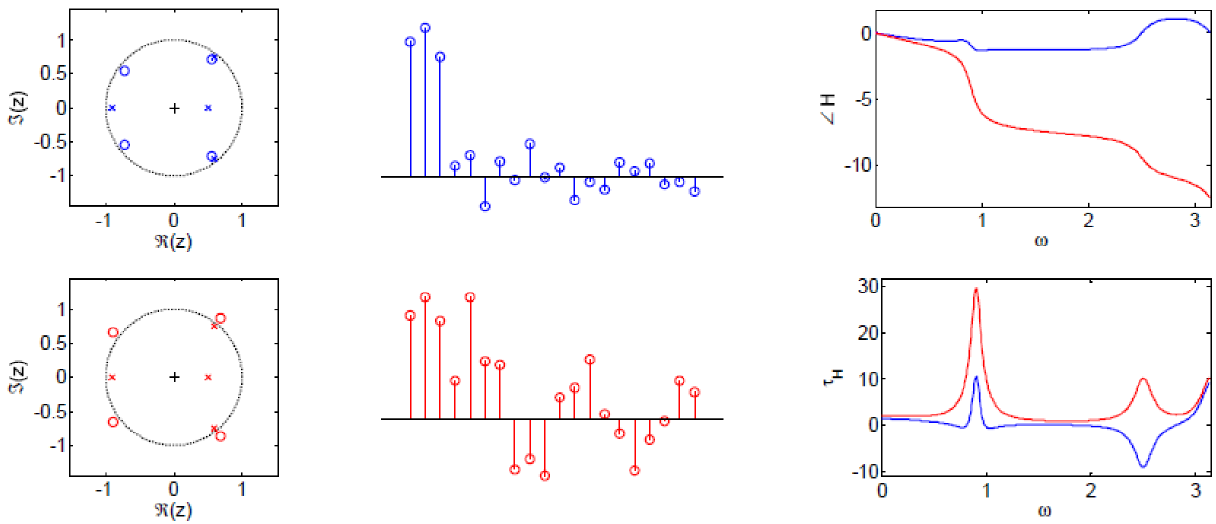
Minimum phase

Average group delay (over ω) = (# poles – # zeros) within the unit circle

- zeros <u>on</u> the unit circle count $-\frac{1}{2}$



Reflecting an interior zero to the exterior multiplies $\left|H(e^{j\omega})\right|$ by a constant but increases average group delay by 1 sample.



A filter with all zeros inside the unit circle is a minimum phase filter:

- Lowest possible group delay for a given magnitude response
- Energy in $h[n]$ is concentrated towards $n = 0$

Energy concentration property

This proof is not examinable

Suppose $H(z)$ has a zero inside the unit circle at $z = z_0$ so that we can write $H(z) = \left(1 - z_0 z^{-1}\right) F(z)$. If we flip this zero outside the unit circle, we can write $G(z) = \left(z^{-1} - z_0^*\right) F(z)$ which has the same magnitude response as $H(z)$.

Taking inverse z-transforms, we can write the corresponding time domain equations:

$$h[n] = f[n] - z_0 f[n-1] \text{ and } g[n] = f[n-1] - z_0^* f[n].$$

Now, defining $f[-1] \triangleq 0$, we sum the energy in the first $K+1$ samples of the impulse response:

$$\begin{aligned}
\sum_{k=0}^{K} |h[k]|^2 &= \sum_{k=0}^{K} |f[k] - z_0 f[k-1]|^2 = \sum_{k=0}^{K} \left(f[k] - z_0 f[k-1]\right)\left(f[k] - z_0 f[k-1]\right)^* \\
&= \sum_{k=0}^{K} |f[k]|^2 - z_0 f[k-1] f^*[k] - z_0^* f^*[k-1] f[k] + |z_0|^2 |f[k-1]|^2 \\
&= \sum_{k=0}^{K} |z_0|^2 |f[k]|^2 - z_0 f[k-1] f^*[k] - z_0^* f^*[k-1] f[k] + |f[k-1]|^2 \\
&\quad + \sum_{k=0}^{K} \left(1 - |z_0|^2\right)\left(|f[k]|^2 - |f[k-1]|^2\right)
\end{aligned}$$

Energy concentration property cont.

So, repeating the previous line,

$$\begin{aligned}\sum_{k=0}^{K} |h[k]|^2 &= \sum_{k=0}^{K} |z_0|^2 \, |f[k]|^2 - z_0 f[k-1] f^*[k] - z_0^* f^*[k-1] f[k] + |f[k-1]|^2 \\ &\quad + \sum_{k=0}^{K} \left(1 - |z_0|^2\right) \left(|f[k]|^2 - |f[k-1]|^2\right) \\ &= \sum_{k=0}^{K} \left(f[k-1] - z_0^* f[k]\right) \left(f[k-1] - z_0^* f[k]\right)^* + \left(1 - |z_0|^2\right) \sum_{k=0}^{K} \left(|f[k]|^2 - |f[k-1]|^2\right) \\ &= \sum_{k=0}^{K} |g[k]|^2 + \left(1 - |z_0|^2\right) \left(|f[K]|^2 - |f[-1]|^2\right) \\ &= \sum_{k=0}^{K} |g[k]|^2 + \left(1 - |z_0|^2\right) |f[K]|^2 \geq \sum_{k=0}^{K} |g[k]|^2\end{aligned}$$

since $|z_0| < 1$ implies that $\left(1 - |z_0|^2\right) > 0$. Thus flipping a zero from inside the unit circle to outside never increases the energy in the first $K + 1$ samples of the impulse response (for any K). Hence the minimum phase response is the one with the most energy in the first $K + 1$ samples for any K.

Linear phase filters

The phase of a linear phase filter is: $\angle H(e^{j\omega}) = \theta_0 - \alpha\omega$

Equivalently constant group delay: $\tau_H = -\frac{d\angle H(e^{j\omega})}{d\omega} = \alpha$

A filter has linear phase iff $h[n]$ is symmetric or antisymmetric:

$$h[n] = h[M-n] \quad \forall n \text{ or else } h[n] = -h[M-n] \quad \forall n$$

M can be even ($\Rightarrow \exists$ mid point) or odd ($\Rightarrow \nexists$ mid point)

Proof $\Leftarrow$:

$$\begin{aligned} 2H(e^{j\omega}) &= \sum_0^M h[n]e^{-j\omega n} + \sum_0^M h[M-n]e^{-j\omega(M-n)} \\ &= e^{-j\omega\frac{M}{2}} \sum_0^M h[n]e^{-j\omega\left(n-\frac{M}{2}\right)} + h[M-n]e^{j\omega\left(n-\frac{M}{2}\right)} \end{aligned}$$

$h[n]$ symmetric:

$$2H(e^{j\omega}) = 2e^{-j\omega\frac{M}{2}} \sum_0^M h[n]\cos\left(n-\frac{M}{2}\right)\omega$$

$h[n]$ anti-symmetric:

$$\begin{aligned} 2H(e^{j\omega}) &= -2je^{-j\omega\frac{M}{2}} \sum_0^M h[n]\sin\left(n-\frac{M}{2}\right)\omega \\ &= 2e^{-j\left(\frac{\pi}{2}+\omega\frac{M}{2}\right)} \sum_0^M h[n]\sin\left(n-\frac{M}{2}\right)\omega \end{aligned}$$

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Summary

- Useful filters have difference equations:
  - Freq response determined by pole/zero positions
  - $N-M$ zeros at origin (or $M-N$ poles)
  - Geometric construction of $|H(e^{j\omega})|$
    - Pole bandwidth $\approx 2\left|\log|p|\right| \approx 2(1-|p|)$
  - Stable if poles have $|p|<1$
- Allpass filter: $a[n]=b[M-n]$
  - Reflecting a zero in unit circle leaves $|H(e^{j\omega})|$ unchanged
- Group delay: $\tau_H\left(e^{j\omega}\right)=-\frac{d\angle H(e^{j\omega})}{d\omega}$ samples
  - Symmetrical $h[n] \Leftrightarrow \tau_H\left(e^{j\omega}\right)=\frac{M}{2}\forall\omega$
  - Average τ_H over $\omega=$ (# poles – # zeros) within the unit circle
- Minimum phase if zeros have $|q|\le 1$
  - Lowest possible group delay for given $|H(e^{j\omega})|$
- Linear phase = Constant group Delay = symmetric/antisymmetric $h[n]$

For further details see Mitra: 6, 7.