Imperial College London

Signals and Systems

Tutorial Sheet 7 – Time and Frequency Response and Filters

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Problem 1 (a)

For a LTI system described by the transfer function

$$H(s) = \frac{s+3}{(s+2)^2}$$

find the system's response (output in time domain) to the input $\cos(2t + 60°)$.

Solution

We proved in lectures that:

$\cos(\omega_0 t + \theta_0) \Rightarrow |H(j\omega_0)|\cos[\omega_0 t + \theta_0 + \angle H(j\omega_0)]$

$H(j\omega) = \frac{j\omega+3}{(j\omega+2)^2}$, $|H(j\omega)| = \frac{\sqrt{\omega^2+9}}{\omega^2+4}$ and $\angle H(j\omega) = \arctan\left(\frac{\omega}{3}\right) - 2\arctan(\frac{\omega}{2})$.

$$|H(j2)| = \frac{\sqrt{2^2+9}}{2^2+4} = \frac{\sqrt{13}}{8}$$

$$\angle H(j2) = \arctan\left(\frac{2}{3}\right) - 2\arctan(1) = 33.69006766 - 2 \cdot 45 = -56.3099$$

$$\cos(2t + 60°) \Rightarrow \frac{\sqrt{13}}{8}\cos[2t + 60° - 56.3099°] = \frac{\sqrt{13}}{8}\cos[2t + 3.69°]$$

Problem 1 (b)

For the previous system find the response to the input $\sin(3t - 45°)$.

Solution

$\sin(\theta) = \cos(\theta - \frac{\pi}{2})$ and therefore, $\sin(3t - 45°) = \cos\left(3t - 45° - \frac{\pi}{2}\right) = \cos(3t - 135°)$.

$\cos(\omega_0 t + \theta_0) \Rightarrow |H(j\omega_0)|\cos[\omega_0 t + \theta_0 + \angle H(j\omega_0)]$

$|H(j\omega)| = \frac{\sqrt{\omega^2+9}}{\omega^2+4}$ and $\angle H(j\omega) = \arctan\left(\frac{\omega}{3}\right) - 2\arctan(\frac{\omega}{2})$.

$$|H(j3)| = \frac{\sqrt{3^2+9}}{3^2+4} = \frac{\sqrt{18}}{13}$$

$$\angle H(j3) = \arctan\left(\frac{3}{3}\right) - 2\arctan\left(\frac{3}{2}\right) = 45 - 2 \cdot 56.30993247°$$
$$= -67.6198648°$$

$\cos(3t - 135°) = \sin(3t - 45°) \Rightarrow \frac{\sqrt{18}}{13}\cos[3t - 135° - 67.6198648°] = \frac{\sqrt{18}}{13}\cos[3t - 45° - 67.6198648° - 90°] = \frac{\sqrt{18}}{13}\sin[3t - 112.6198°]$

Problem 1 (c)

For the previous system find the system's response to the input e^{j3t}.

Solution

$e^{j3t} = \cos(3t) + j\sin(3t)$ with $\sin(3t) = \cos\left(3t - \frac{\pi}{2}\right) = \cos(3t - 90°)$.

$\cos(\omega_0 t + \theta_0) \Rightarrow |H(j\omega_0)|\cos[\omega_0 t + \theta_0 + \angle H(j\omega_0)]$

$$|H(j3)| = \frac{\sqrt{3^2+9}}{3^2+4} = \frac{\sqrt{18}}{13}$$

$\angle H(j3) = -67.6198648°$

$$\cos(3t) \Rightarrow \frac{\sqrt{18}}{13}\cos[3t - 67.6198648°]$$

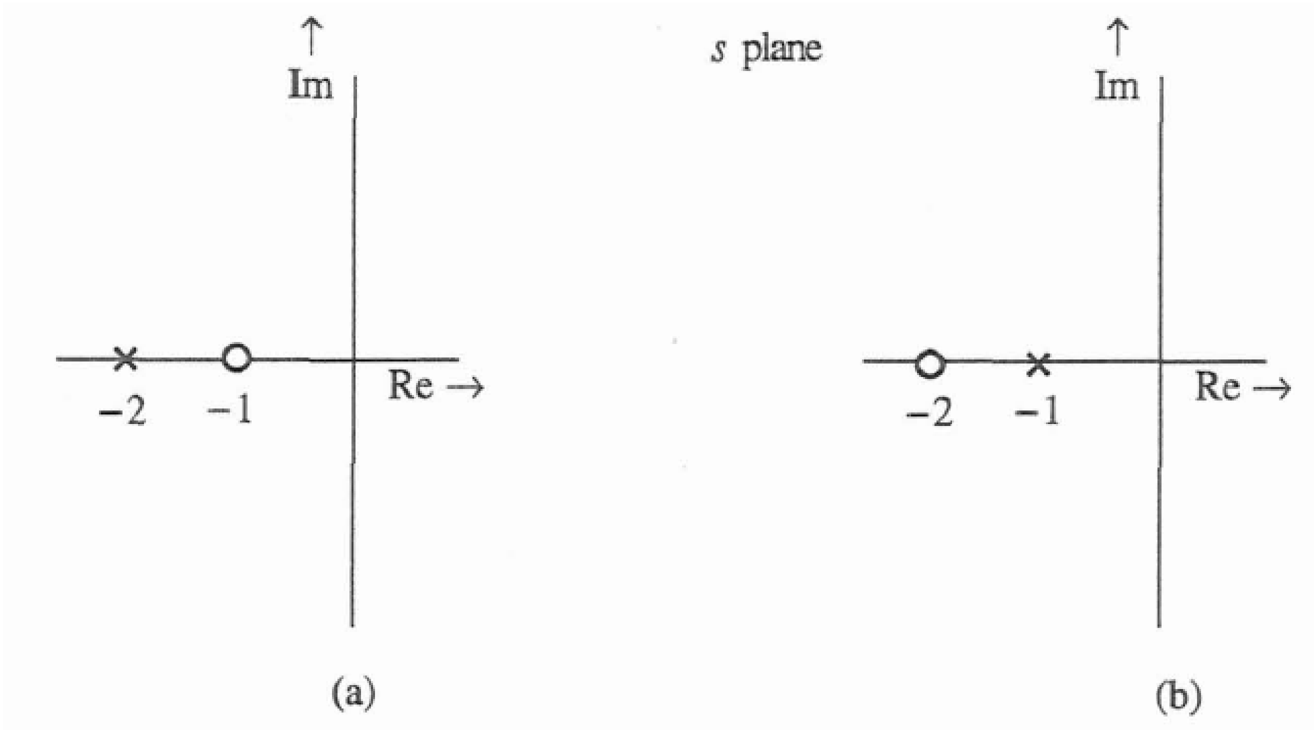
$$\cos(3t - 90°) \Rightarrow \frac{\sqrt{18}}{13}\cos[3t - 67.6198648° - 90°] = \frac{\sqrt{18}}{13}\sin[3t - 67.6198648°]$$

$$e^{j3t} = \cos(3t) + j\sin(3t) \Rightarrow \frac{\sqrt{18}}{13}\cos[3t - 67.6198648°] + j\frac{\sqrt{18}}{13}\sin[3t - 67.6198648°] = \frac{\sqrt{18}}{13}\, e^{j(3t-67.6198648°)t} = e^{j3t}\,|H(j3)|\, e^{j\angle H(j3)} = e^{j3t}\, H(j3)$$

(this result could be used directly, since it has also been proven in lectures)

Problem 2

Draw a rough sketch of the amplitude and phase response of the LTI systems whose pole-zero plots are shown in Figures (a) and (b) below.



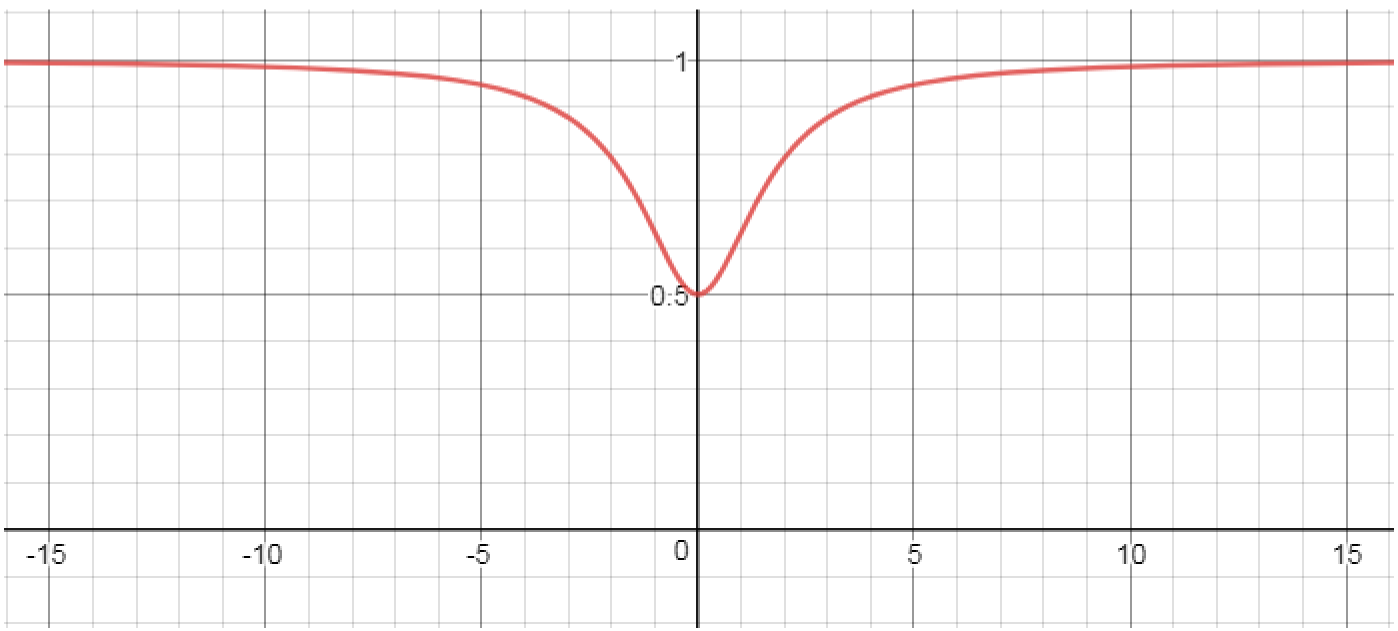
Problem 2 (a)

We have a zero at -1 and a pole at -2. Therefore, $H(s) = \frac{s+1}{s+2}$

$H(j\omega) = \frac{j\omega+1}{j\omega+2}$. The amplitude is $|H(j\omega)| = \frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+4}}$ with $|H(j\omega)|^2 = \frac{\omega^2+1}{\omega^2+4}$.

$\frac{d}{d\omega}|H(j\omega)|^2 = \frac{2\omega(\omega^2+4)-2\omega(\omega^2+1)}{(\omega^2+4)^2} > 0$. Therefore, $|H(j\omega)|^2$ is an increasing function.

$|H(j0)| = 1/2$ and $|H(\infty)| = 1$.



Problem 2 (a) cont.

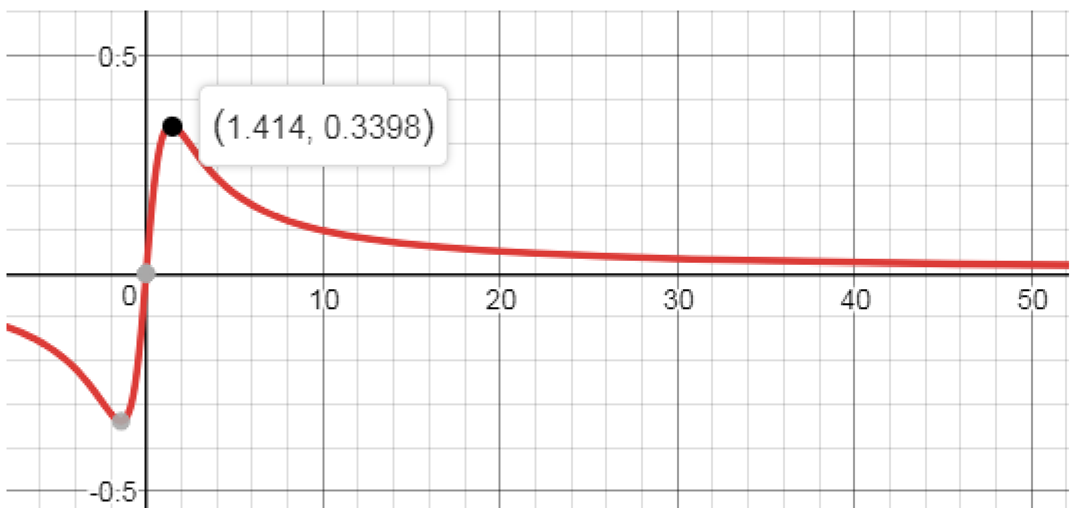
$H(j\omega) = \frac{j\omega+1}{j\omega+2}$ and the phase is $\angle H(j\omega) = \arctan(\omega) - \arctan(\frac{\omega}{2})$.

We see that $\angle H(j0) = 0$ and $\angle H(j\omega) = 0$.

$$\frac{d}{d\omega}\angle H(j\omega) = \frac{1}{1+\omega^2} - \frac{1}{2}\frac{1}{1+\frac{\omega^2}{4}} = \frac{2+\frac{\omega^2}{2}-(1+\omega^2)}{2(1+\omega^2)\left(1+\frac{\omega^2}{4}\right)} = \frac{1-\frac{\omega^2}{2}}{2(1+\omega^2)\left(1+\frac{\omega^2}{4}\right)}$$

For $\frac{\omega^2}{2} < 1 \Rightarrow \omega^2 < 2 \Rightarrow \omega < \sqrt{2} = 1.414$ the phase is increasing.

For $\frac{\omega^2}{2} > 1 \Rightarrow \omega > \sqrt{2} = 1.414$ the phase is decreasing.



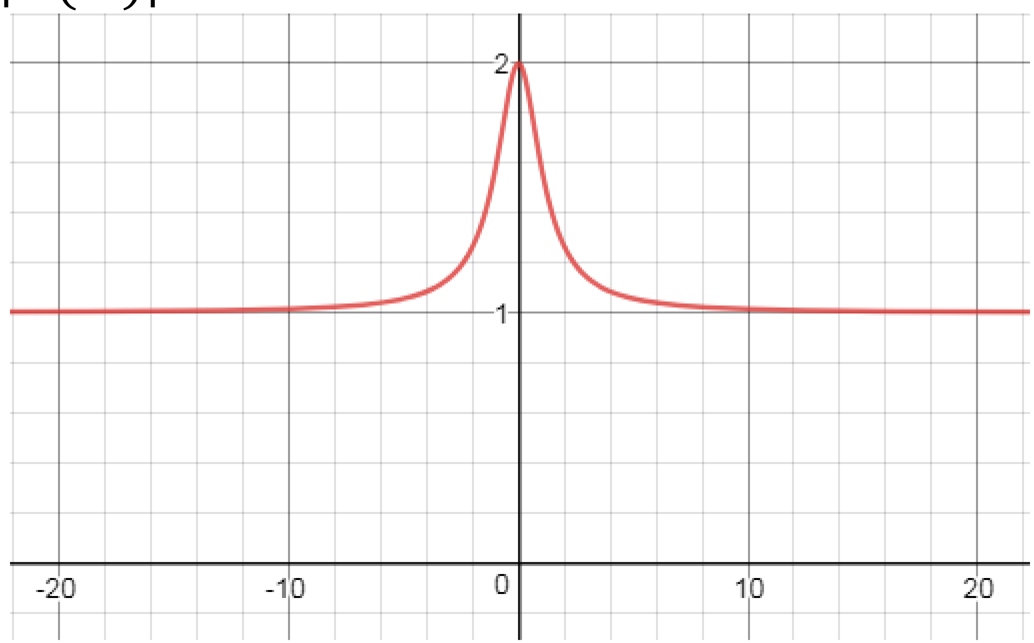
Problem 2 (b)

We have a zero at -2 and a pole at -1. Therefore, $H(s) = \frac{s+2}{s+1}$.

$H(j\omega) = \frac{j\omega+2}{j\omega+1}$. The amplitude is $|H(j\omega)| = \frac{\sqrt{\omega^2+4}}{\sqrt{\omega^2+1}}$ with $|H(j\omega)|^2 = \frac{\omega^2+4}{\omega^2+1}$.

$\frac{d}{d\omega}|H(j\omega)|^2 = \frac{2\omega(\omega^2+1)-2\omega(\omega^2+4)}{(\omega^2+1)^2} < 0.$ Therefore, $|H(j\omega)|^2$ is an decreasing function.

$|H(j0)| = 2$ and $|H(\infty)| = 1$.



Problem 2 (b) cont.

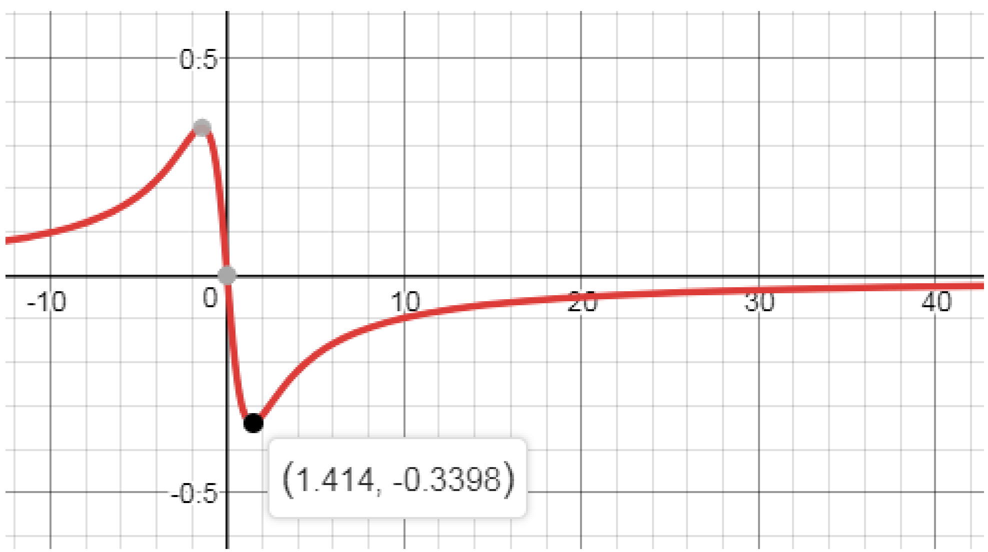
$H(j\omega) = \frac{j\omega+2}{j\omega+1}$ and the phase is $\angle H(j\omega) = \arctan(\omega/2) - \arctan(\omega)$.

We see that $\angle H(j0) = 0$ and $\angle H(j\omega) = 0$.

$$\frac{d}{d\omega}\angle H(j\omega) = -\frac{1}{1+\omega^2} + \frac{1}{2}\frac{1}{1+\frac{\omega^2}{4}} = \frac{-2-\frac{\omega^2}{2}+(1+\omega^2)}{2(1+\omega^2)\left(1+\frac{\omega^2}{4}\right)} = \frac{\frac{\omega^2}{2}-1}{2(1+\omega^2)\left(1+\frac{\omega^2}{4}\right)}$$

For $\frac{\omega^2}{2} < 1 \Rightarrow \omega^2 < 2 \Rightarrow \omega < \sqrt{2} = 1.414$ the phase is decreasing.

For $\frac{\omega^2}{2} > 1 \Rightarrow \omega > \sqrt{2} = 1.414$ the phase is increasing.

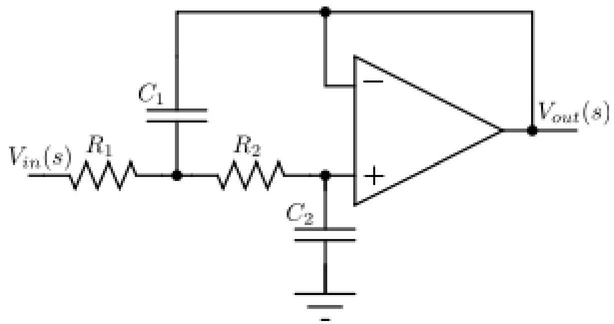


Problem 3: Design of Butterworth filters with Sallen-Key

- The transcription function of the Sallen-Key filter on the right is:

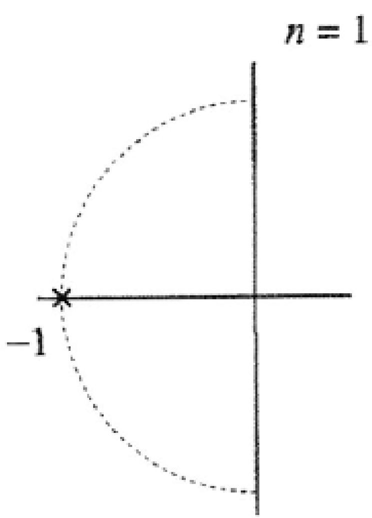
$$H(s) = \frac{1}{1 + C_2(R_1 + R_2)s + C_1 C_2 R_1 R_2 s^2}$$

- Assuming that $\omega_c = 1$ and n even we choose:
  - $C_1 C_2 R_1 R_2 = 1$
  - $C_2(R_1 + R_2) = -2\cos(\frac{2k+n-1}{2n}\pi)$

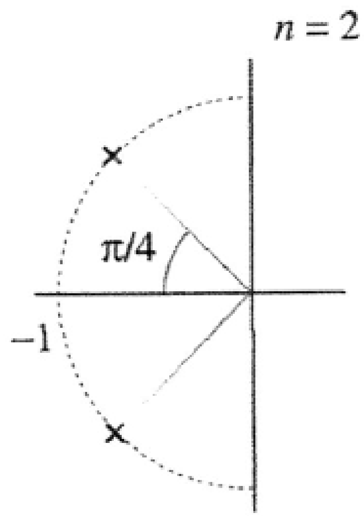


- This guarantees that $H(s)$ has two poles at $\cos(\frac{2k+n-1}{2n}\pi) \pm j\sin\left(\frac{2k+n-1}{2n}\pi\right)$.
- Cascade $n/2$ such filters.
- When n is odd the remaining real pole can be implemented with an RC circuit.

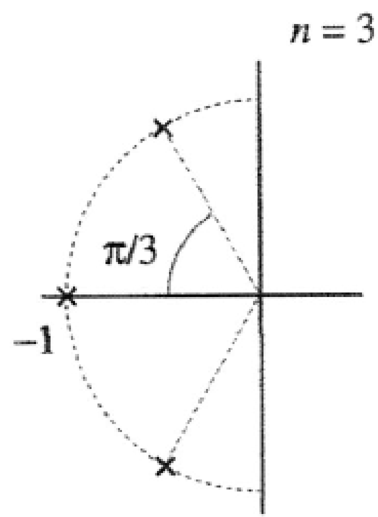
Problem 3: Design of Butterworth filters with Sallen-Key cont.



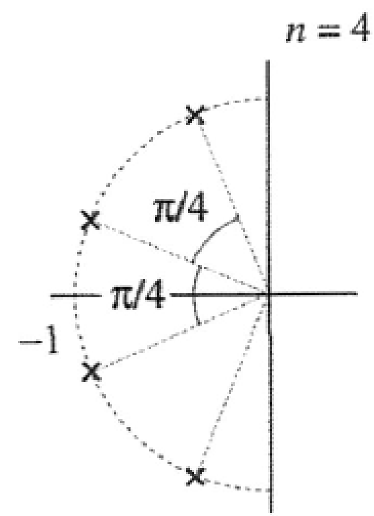
One RC circuit



One Sallen-Key with $k = 1$ and $n = 2$



One Sallen-key with $k = 1$, $n = 3$ followed by one RC circuit



A cascade of two Sallen-key with $n = 4$ and $k = 1, 2$.

Problem 3: Design of Butterworth filters with Sallen-Key cont.

- So far we have considered only normalized Butterworth filters with $3dB$ bandwidth and $\omega_c = 1$.
- We can design filters for any other cut-off frequency by substituting s by s/ω_c.
- For example, the transfer function for a second-order Butterworth filter for $\omega_c = 100$ is given by:

$$H(s) = \frac{1}{(\frac{s}{100})^2 + \sqrt{2}\left(\frac{s}{100}\right) + 1} = \frac{1}{s^2 + 100\sqrt{2}s + 10^4}$$