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# Signals and Systems

## Tutorial Sheet 4 – Laplace Transforms

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# Problem 1 (a)

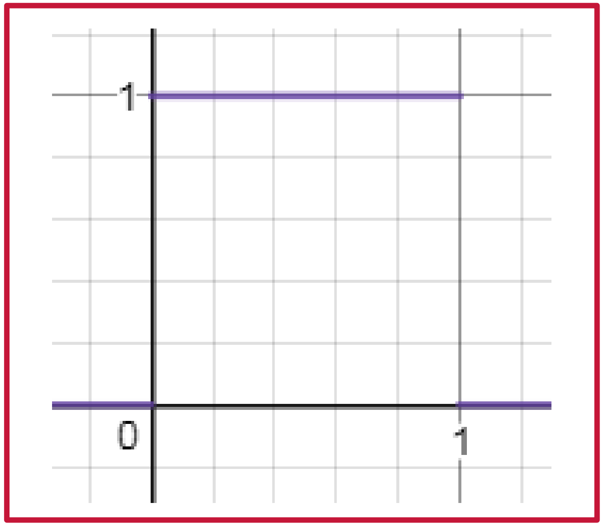
By direct integration, find the Laplace transforms of the following functions.

(a) $f(t) = u(t) - u(t-1)$, $u(t)$ is the unit step function.

By definition we have:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty}\big(u(t) - u(t-1)\big)e^{-st}dt = \int_0^1 e^{-st}dt = -\frac{e^{-st}}{s}\bigg|_0^1 =$$

$$= -\frac{1}{s}(e^{-s} - 1) = \frac{1}{s}(1 - e^{-s})$$



# Problem 1 (b)

(b) $f(t) = te^{-t}u(t)$

$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} te^{-t}u(t)e^{-st}dt = \int_{0}^{+\infty} te^{-t}e^{-st}dt = \int_{0}^{+\infty} te^{-(s+1)t}dt$

For this integral we might use **partial integration** as follows:

$$\int_{0}^{+\infty} te^{-(s+1)t}dt = -\frac{1}{(s+1)}\int_{0}^{+\infty} td(e^{-(s+1)t})$$

$$= -\frac{1}{(s+1)}te^{-(s+1)t}\Big|_0^{+\infty} + \frac{1}{(s+1)}\int_0^{+\infty} e^{-(s+1)t}dt$$


$$= -\frac{1}{(s+1)}te^{-(s+1)t}\Big|_0^{+\infty} - \frac{1}{(s+1)^2}e^{-(s+1)t}\Big|_0^{+\infty}$$

If $\mathcal{Re}\{s+1\} > 0$ then $\lim_{t\to+\infty} te^{-(s+1)t} = 0$ (because $e^{-(s+1)t}$ goes to $-\infty$ faster than $t$ going to $+\infty$) and $\lim_{t\to+\infty} e^{-(s+1)t} = 0$, therefore,

$$-\frac{1}{(s+1)}te^{-(s+1)t}\Big|_0^{+\infty} = \lim_{t\to+\infty} -\frac{1}{(s+1)}te^{-(s+1)t} + \frac{1}{(s+1)}0\cdot e^{-(s+1)\cdot 0} = 0$$

$$-\frac{1}{(s+1)^2}e^{-(s+1)t}\Big|_0^{+\infty} = \lim_{t\to+\infty} -\frac{1}{(s+1)^2}e^{-(s+1)t} + \frac{1}{(s+1)^2}\cdot e^{-(s+1)\cdot 0} = \frac{1}{(s+1)^2}$$

$\int_0^{+\infty} te^{-(s+1)t}dt = \frac{1}{(s+1)^2}$ with ROC $\mathcal{Re}\{s+1\} > 0$.

# Problem 1 (c)

(c) $f(t) = t\cos(\omega_0 t)u(t)$

Recall from the lecture that if

$x(t) = \cos(\omega_0 t)u(t)$ then

$$\mathcal{L}\{x(t)\} = \frac{1}{2}\left[\frac{1}{(s-j\omega_0)} + \frac{1}{(s+j\omega_0)}\right] = \frac{s}{s^2+{\omega_0}^2}, \quad \mathrm{Re}\{s\} > 0$$

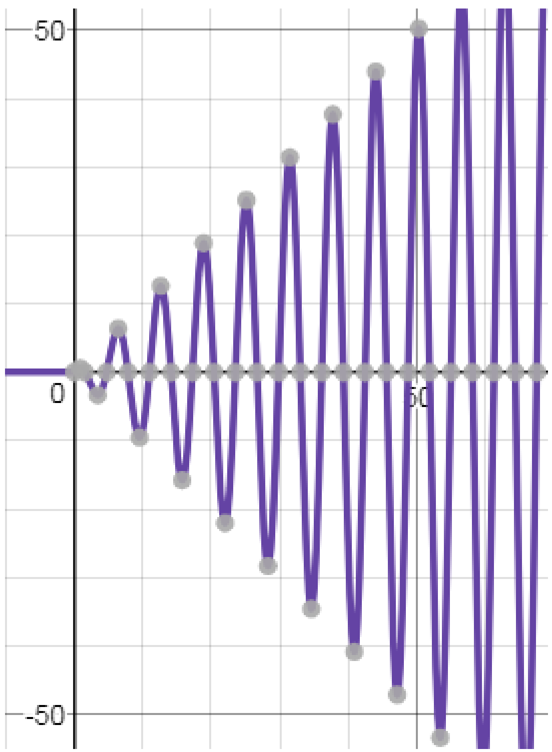
By using the frequency differentiation property which states that

$$\mathcal{L}\{tx(t)\} = -\frac{dX(s)}{ds}$$

we see that

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{tx(t)\} = -\frac{d}{ds}\left(\frac{\mathbf{s}}{\mathbf{s}^2+\boldsymbol{\omega_0}^2}\right)$$

$$= -\frac{s^2+{\omega_0}^2-2s^2}{(s^2+{\omega_0}^2)^2} = \frac{s^2-{\omega_0}^2}{(s^2+{\omega_0}^2)^2}$$




$\boldsymbol{\omega_0} = \mathbf{1}$ **in the Figure**

# Problem 1 (d)

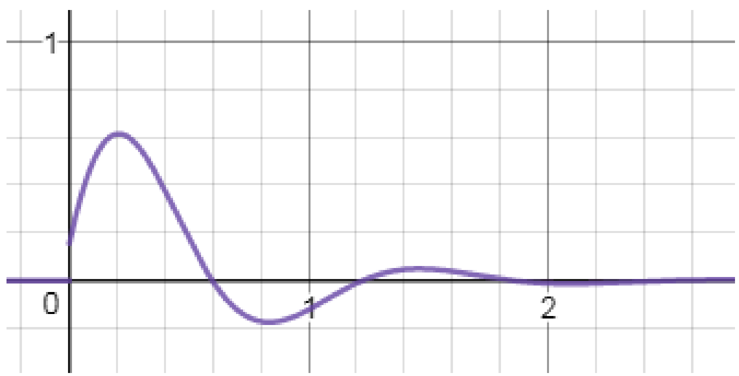
(d) $f(t) = e^{-2t}\cos(5t+\theta)u(t)$

$f(t) = \frac{1}{2}e^{-2t}(e^{j(5t+\theta)}+e^{-j(5t+\theta)})u(t)$

$= (\frac{1}{2}e^{j\theta}e^{-(2-j5)t} + \frac{1}{2}e^{-j\theta}e^{-(2+j5)t})u(t)$

$F(s) = \frac{1}{2}e^{j\theta}\frac{1}{s+2-j5} + \frac{1}{2}e^{-j\theta}\frac{1}{s+2+j5}$

$= \frac{1}{2}\frac{e^{j\theta}(s+2+j5)+e^{-j\theta}(s+2-j5)}{(s+2)^2+25}$



$\theta = 30^o$ in the Figure

We see that:

$e^{j\theta}j - e^{-j\theta}j = j\big(\cos(\theta) + j\sin(\theta)\big) - j\big(\cos(\theta) - j\sin(\theta)\big) = -2\sin(\theta)$

Therefore,

$$F(s) = \frac{1}{2}\frac{(s+2)(e^{j\theta}+e^{-j\theta}) + 5\,(e^{j\theta}j - e^{-j\theta}j)}{(s+2)^2+25} =$$

$$= \frac{1}{2}\frac{(s+2)2\cos(\theta)+5\,(-2\sin(\theta)\,)}{(s+2)^2+25} = \frac{(s+2)\cos(\theta)-5\sin(\theta)}{s^2+4s+29}$$

# Problem 2 (a)

By direct integration, find the Laplace transforms of the following signals.

(a) $f(t) = t(u(t) - u(t-1))$, $u(t)$ is the unit step function.

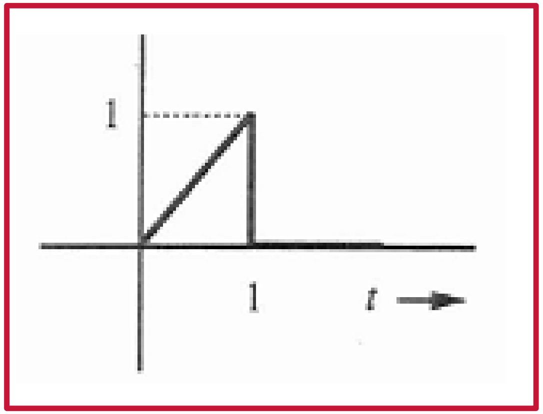
$F(s) = \int_{-\infty}^{+\infty} t\big(u(t) - u(t-1)\big)e^{-st}dt = \int_0^1 te^{-st}dt$

For this integral we might use **partial integration** as follows:

$$\int_0^1 te^{-st}dt = -\frac{1}{s}\int_0^1 td(e^{-st})$$

$$= -\frac{1}{s}te^{-st}|_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt = -\frac{1}{s}te^{-st}|_0^1 - \frac{1}{s^2}e^{-st}|_0^1$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2} = \frac{1}{s^2}(1 - e^{-s} - se^{-s})$$



# Problem 2 (b)

(b) $f(t) = \sin(t)(u(t) - u(t-\pi))$, $u(t)$ is the unit step function.

$F(s) = \int_{-\infty}^{+\infty} \sin(t)\big(u(t) - u(t-\pi)\big)e^{-st}dt = \int_0^{\pi} \sin(t)e^{-st}dt$

We use the following relationship from integral tables:

(https://en.wikipedia.org/wiki/List_of_integrals_of_exponential_functions)

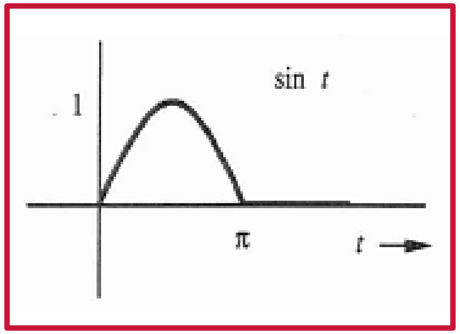
$$\int \sin(bt)e^{at}dt = \frac{e^{at}}{a^2+b^2}\big(a\sin(bt) - b\cos(bt)\big)$$

In this particular example we have: $b = 1$, $a = -s$

$$F(s) = \int_0^{\pi} \sin(t)e^{-st}dt = \frac{e^{-st}}{s^2+1}\big(-s \cdot \sin(t) - \cos(t)\big)\Big|_0^{\pi}$$

$$= \frac{e^{-s\pi}}{s^2+1}\big(-s \cdot \sin(\pi) - \cos(\pi)\big) - \frac{e^{-s\cdot 0}}{s^2+1}\big(-s \cdot \sin(0) - \cos(0)\big)$$

$$= \frac{e^{-s\pi}}{s^2+1}(0 - (-1)) - \frac{1}{s^2+1}(0 - 1) = \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} = \frac{e^{-s\pi}+1}{s^2+1}$$



# Problem 2 (c)

(c) $f(t) = \frac{1}{e}t\big(u(t) - u(t-1)\big) + e^{-t}u(t-1)$

$$F(s) = \int_{-\infty}^{+\infty} \frac{1}{e}t\big(u(t) - u(t-1)\big)e^{-st}dt + \int_{-\infty}^{+\infty} e^{-t}u(t-1)e^{-st}dt$$

For the second term we have:

$\int_{-\infty}^{+\infty} e^{-t}u(t-1)e^{-st}dt = \int_{1}^{+\infty} e^{-t}e^{-st}dt = \int_{1}^{+\infty} e^{-(s+1)t}dt = \frac{1}{-(s+1)}e^{-(s+1)t}\Big|_{1}^{+\infty}$

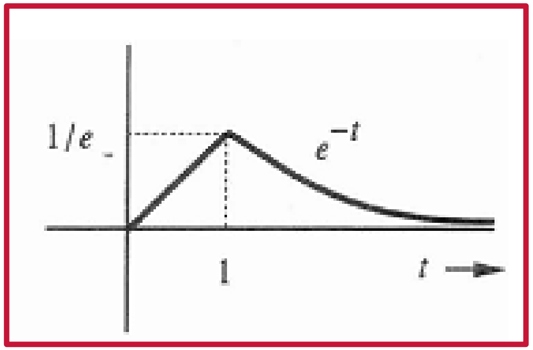
$\lim_{t\to+\infty} -\frac{1}{s+1}e^{-(s+1)t} = 0$ if $\mathcal{R}e\{s+1\} > 0$

Second term: $\int_{-\infty}^{+\infty} e^{-t}u(t-1)e^{-st}dt = \frac{1}{(s+1)}e^{-(s+1)}$

First term: $\int_{-\infty}^{+\infty} \frac{1}{e}t\big(u(t) - u(t-1)\big)e^{-st}dt = \frac{1}{es^2}(1 - e^{-s} - se^{-s})$

(see Problem 2(a))

Therefore, $F(s) = \frac{1}{es^2}(1 - e^{-s} - se^{-s}) + \frac{1}{(s+1)}e^{-(s+1)}$



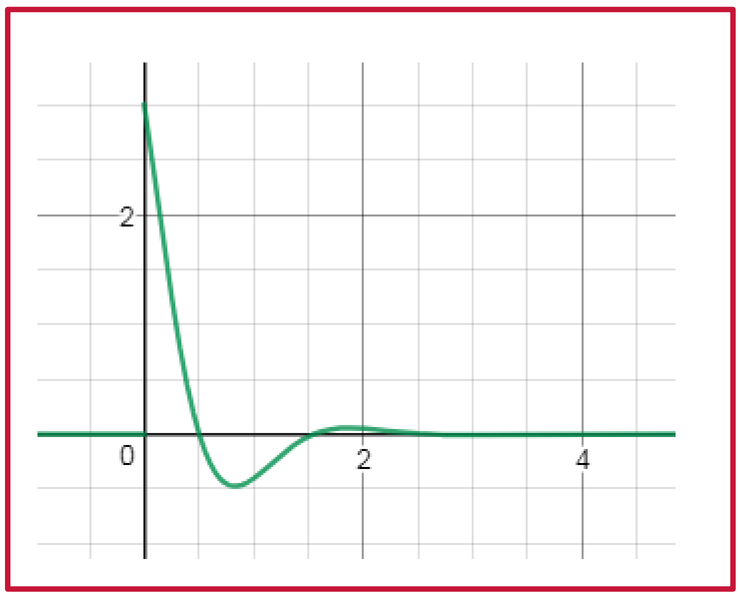
# Problem 3 (a)

Find the inverse Laplace transforms of the following functions, assuming that the corresponding function in time is causal.

(a) $F(s) = \frac{2s+5}{s^2+5s+6}$

$$F(s) = \frac{2s+5}{s^2+5s+6} = \frac{(s+3)+(s+2)}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$

$$f(t) = \mathcal{L}^{-1}\big(F(s)\big) = (e^{-2t} + e^{-3t})u(t)$$



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# Problem 3 (b)

(b) $F(s) = \frac{3s+5}{s^2+4s+13}$

For this function we can use the Laplace transform pair:

$\mathcal{L}\{re^{-at}\cos(bt+\theta)\,u(t)\} = \frac{As+B}{s^2+2as+c}$ (Look at Property 10c Slide 28)

$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $\theta = \tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$, $b = \sqrt{c-a^2}$

$A = 3, B = 5, a = 2, c = 13$, therefore,

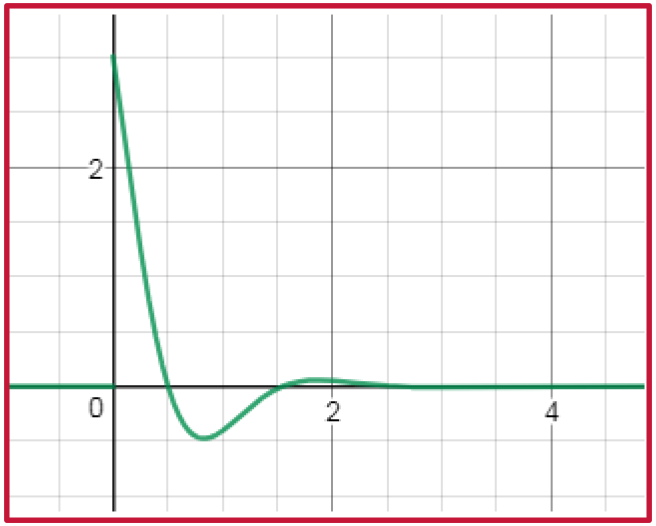
$$r = \sqrt{\frac{3^2 \cdot 13+5^2-2\cdot 3\cdot 5\cdot 2}{13-2^2}} = \sqrt{\frac{117+25-60}{9}} = \sqrt{\frac{82}{9}} = 3.018$$

$$\theta = \tan^{-1}\left(\frac{3\cdot 2-5}{3\sqrt{13-2^2}}\right) = \tan^{-1}\left(\frac{1}{9}\right) = 6.34^o$$

$$b = \sqrt{13-2^2} = 3$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{As+B}{s^2+2as+c}\right)$$

$$= 3.018e^{-2t}\cos(3t+6.34^o)\,u(t)$$



# Problem 3 (c)

(c) $F(s) = \frac{(s+1)^2}{s^2-s-6}$

The power of the numerator is the same as the power of the denominator. In that case, in the partial fraction expansion we have to add the coefficient of the highest power of the numerator to it.

(Recall Lecture 6-7)

$$F(s) = \frac{(s+1)^2}{s^2-s-6} = \frac{s^2+2s+1}{(s+2)(s-3)} = 1 + \frac{A}{(s+2)} + \frac{B}{(s-3)}$$

$$= \frac{s^2-s-6+A(s-3)+B(s+2)}{(s+2)(s-3)} = \frac{s^2+(A+B-1)s+(-3A+2B-6)}{(s+2)(s-3)}$$

$A + B - 1 = 2 \Rightarrow 3A + 3B = 9$ (1)

$-3A + 2B - 6 = 1 \Rightarrow -3A + 2B = 7$ (2)

$5B = 16 \Rightarrow B = 16/5 = 3.2$ and $A = -1/5 = -0.2$

$$F(s) = \frac{(s+1)^2}{s^2-s-6} = 1 - \frac{0.2}{(s+2)} + \frac{3.2}{(s-3)}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$= \delta(t) + (3.2e^{3t} - 0.2e^{-2t})u(t)$

# Problem 3 (d)

(d) $F(s) = \frac{2s+1}{(s+1)(s^2+2s+2)}$

Partial fraction expansion in that case gives:

$$F(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+2s+2)}$$

$A + B = 0 \Rightarrow A = -B$ (1)

$2A + B + C = 2 \Rightarrow -2B + B + C = 2$ (2)

$2A + C = 1 \Rightarrow -2B + C = 1$ (3)

From (2) and (3) we have $B = 1$. Therefore, $A = -1$ and $C = 3$.

$$F(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{-1}{(s+1)} + \frac{s+3}{(s^2+2s+2)}$$

For the first term we have: $\mathcal{L}^{-1}\left\{\frac{-1}{(s+1)}\right\} = -e^{-t}u(t)$

For the second term we use the Laplace transform pair that we used in Problem 3(b). Please see next slide.

# Problem 3 (d) cont.

(d) $\frac{s+3}{s^2+2s+2}$

For this function we can use the Laplace transform pair:

$$\mathcal{L}\{re^{-at}\cos(bt+\theta)\,u(t)\} = \frac{As+B}{s^2+2as+c}$$

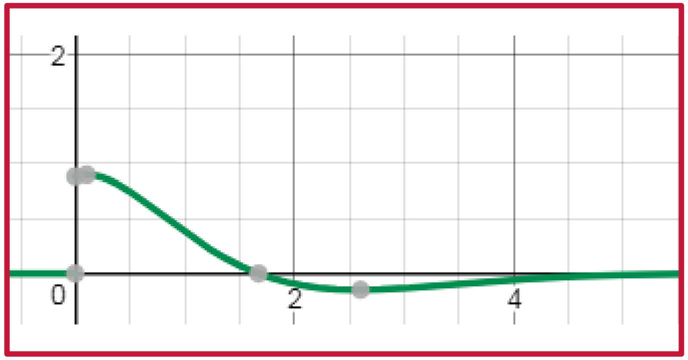
$$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}},\ \theta = \tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right),\ b = \sqrt{c-a^2}$$

$A = 1, B = 3, a = 1, c = 2$, therefore,

$$r = \sqrt{\frac{1^2\cdot2+3^2-2\cdot1\cdot3\cdot1}{2-1^2}} = \sqrt{\frac{2+9-6}{1}} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{1\cdot1-3}{1\sqrt{2-1}}\right) = \tan^{-1}(-2) = 63.4^o$$

$$b = \sqrt{2-1^2} = 1$$



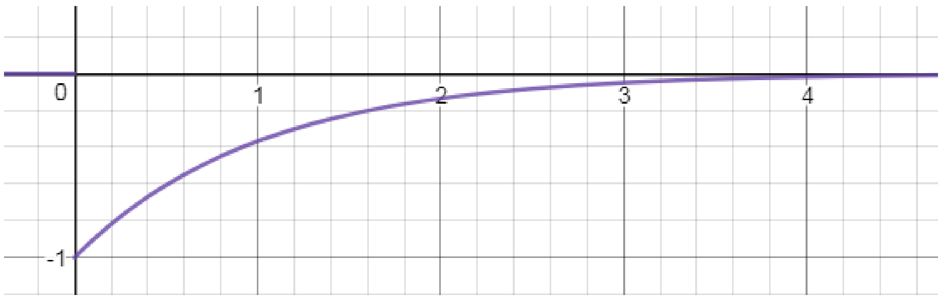

$$\mathcal{L}^{-1}\left(\frac{s+3}{s^2+2s+2}\right) = \sqrt{5}e^{-t}\cos(t-63.4^o)\,u(t)$$

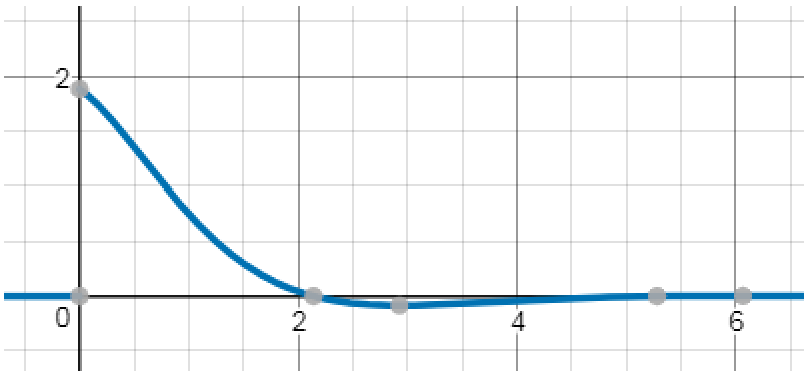
From the two previous functions we have:

$$f(t) = \mathcal{L}^{-1}(F(s)) = (-e^{-t}+\sqrt{5}e^{-t}\cos(t-63.4^o))u(t)$$

# Problem 3 (d) cont.

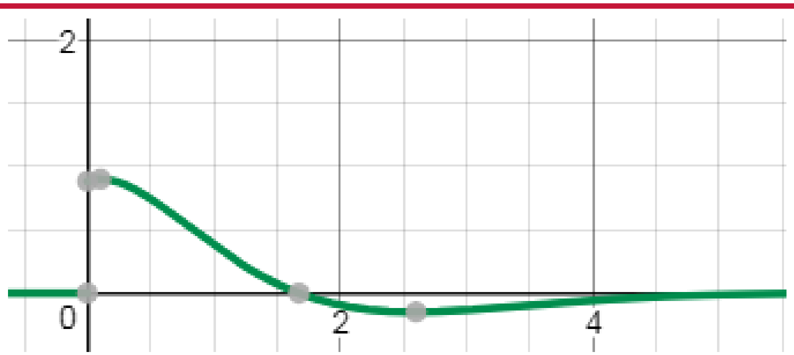
(d) The functions $-e^{-t}u(t)$ and $\sqrt{5}e^{-t}\cos(t-63.4^{o})u(t)$ are depicted below.




The required function is the sum of the two functions above.

$f(t)=\mathcal{L}^{-1}(F(s))=(-e^{-t}+\sqrt{5}e^{-t}\cos(t-63.4^{o}))u(t)$



# Problem 3 (e)

(e) $\frac{s+3}{(s+2)(s+1)^2}$

In that case we have repeated roots, i.e., there is a factor $(s+1)^2$ in the denominator.

In general, if there is a factor $(s+a)^n$ in the denominator, the partial fraction expansion contains the term $\sum_{i=1}^{n} \frac{c_i}{(s+a)^i}$.

$$\frac{s+3}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{A(s+1)^2+B(s+1)(s+2)+C(s+2)}{(s+2)(s+1)^2}$$

$$A + B = 0 \Rightarrow A = -B \qquad (1)$$

$$2A + 3B + C = 1 \Rightarrow -2B + 3B + C = 1 \Rightarrow B + C = 1 \qquad (2)$$

$$A + 2B + 2C = 3 \Rightarrow -B + 2B + 2C = 3 \Rightarrow B + 2C = 3 \qquad (3)$$

$$A = 1, B = -1, C = 2$$

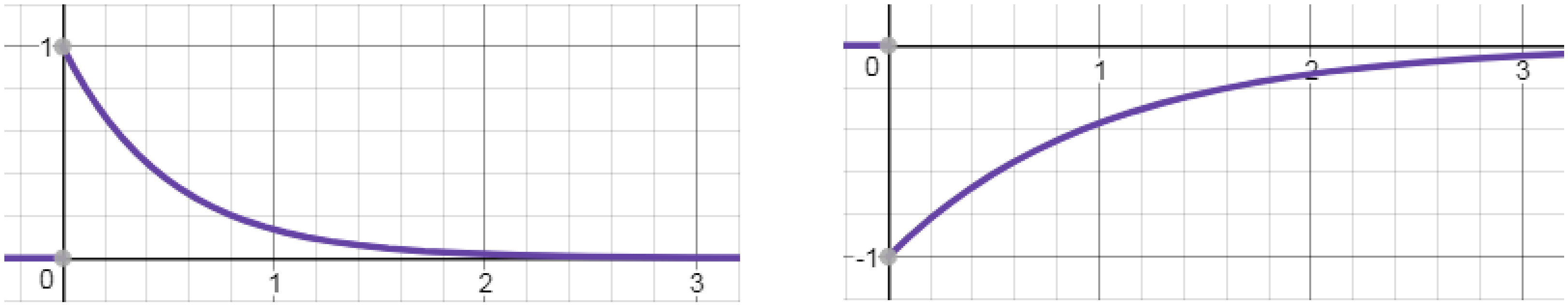
$$F(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{1}{s+2} - \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = (e^{-2t} - e^{-t} + 2te^{-t})u(t)$$

**For the last term refer to Question 1(b).**

# Problem 3 (e) cont.

(e) The functions $e^{-2t}u(t)$ and $-e^{-t}u(t)$ are depicted below, left and right respectively.




The function $2te^{-t}u(t)$ is depicted below.

# Problem 3 (e) cont.

(e) The sum of the above 3 functions is the function $f(t) = (e^{-2t} - e^{-t} + 2te^{-t})u(t)$ depicted below.

# Problem 4 (a), (b)

Find the Laplace transforms of the following function using the Laplace Transform Table and the time−shifting property where appropriate.

(a) $f(t) = u(t) - u(t-1)$, $u(t)$ is the unit step function.

$$F(s) = \mathcal{L}\{u(t) - u(t-1)\} = \mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-1)\}$$

$$= \frac{1}{s} - e^{-s}\frac{1}{s} = \frac{1}{s}(1 - e^{-s})$$

[Recall that the time-shifting property is as follows: $\mathcal{L}\{x(t-t_0)\} = e^{-st_0}X(s)$]

(b) $f(t) = e^{-(t-\tau)}u(t)$, $u(t)$ is the unit step function.

$$f(t) = e^{-(t-\tau)}u(t) = e^{\tau}\, e^{-t}u(t)$$

$$F(s) = e^{\tau}\, \mathcal{L}\{e^{-t}u(t)\} = e^{\tau}\, \frac{1}{s+1}$$

# Problem 4(c)

(c) $f(t) = e^{-t}u(t-\tau)$, $u(t)$ is the unit step function.

$f(t) = e^{-t}u(t-\tau) = e^{-\tau}\, e^{-(t-\tau)}u(t-\tau)$

$$F(s) = e^{-\tau}\,\mathcal{L}\{\mathrm{e}^{-(t-\tau)}u(t-\tau)\} = e^{-\tau}\,e^{-s\tau}\frac{1}{s+1} = e^{-(s+1)\tau}\frac{1}{s+1}$$

Recall that:

$\mathcal{L}\{e^{-t}u(t)\}= \frac{1}{s+1}$

# Problem 4(d)

(d) $f(t) = \sin\big(\omega_0(t-\tau)\big)u(t-\tau)$

Assume that $x(t) = \sin(\omega_0 t)u(t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})u(t)$.

$$\int_{-\infty}^{+\infty} e^{j\omega_0 t}u(t)e^{-st}dt = \int_0^{+\infty} e^{j\omega_0 t}e^{-st}dt = \int_0^{+\infty} e^{-(s-j\omega_0)t}dt = \frac{-1}{(s-j\omega_0)}e^{-(s-j\omega_0)t}\Big|_0^{+\infty}$$

$$= \frac{-1}{(s-j\omega_0)}[(e^{-(s-j\omega_0)\cdot+\infty}) - (e^{-(s-j\omega_0)\cdot 0})] = \frac{-1}{(s-j\omega_0)}(0-1) = \frac{1}{(s-j\omega_0)},\ \mathrm{Re}\{s\} > 0$$

$$\int_{-\infty}^{+\infty} e^{-j\omega_0 t}u(t)e^{-st}dt = \int_0^{+\infty} e^{-j\omega_0 t}e^{-st}dt = \int_0^{+\infty} e^{-(s+j\omega_0)t}dt = \frac{-1}{(s+j\omega_0)}e^{-(s+j\omega_0)t}\Big|_0^{+\infty}$$

$$= \frac{-1}{(s+j\omega_0)}[(e^{-(s+j\omega_0)\cdot+\infty}) - (e^{-(s+j\omega_0)\cdot 0})] = \frac{1}{(s+j\omega_0)}(0-1) = \frac{1}{(s+j\omega_0)},\ \mathrm{Re}\{s\} > 0$$

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# Problem 4(d) cont.

Based on the analysis of the previous slide we have:

$$x(t) = \sin(\omega_0 t)u(t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})u(t).$$

$$\mathcal{L}\{x(t)\} = \mathcal{L}\left\{\frac{1}{2j}e^{j\omega_0 t}u(t)\right\} - \mathcal{L}\left\{\frac{1}{2j}e^{-j\omega_0 t}u(t)\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{(s - j\omega_0)} - \frac{1}{(s + j\omega_0)}\right] = \frac{\omega_0}{s^2 + \omega_0{}^2}, \quad \mathrm{Re}\{s\} > 0$$

$$f(t) = x\,(t - \tau).$$

Therefore, by applying the shifting property we obtain:

$$F(s) = \mathcal{L}\{f(t)\} = \frac{\omega_0}{s^2 + \omega_0{}^2}e^{-\tau s}$$

# Problem 4(e)

(e) $f(t) = \sin\big(\omega_0(t-\tau)\big)u(t) = (\sin(\omega_0 t)\cos(\omega_0\tau) - \cos(\omega_0 t)\sin(\omega_0\tau))u(t)$

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{(\cos(\omega_0\tau)\sin(\omega_0 t)u(t) - \sin(\omega_0\tau)\cos(\omega_0 t))u(t)\}$

$= \mathcal{L}\{\cos(\omega_0\tau)\sin(\omega_0 t)u(t)\} - \mathcal{L}\{\sin(\omega_0\tau)\cos(\omega_0 t)\,u(t)\}$

$= \cos(\omega_0\tau)\,\mathcal{L}\{\sin(\omega_0 t)u(t)\} - \sin(\omega_0\tau)\,\mathcal{L}\{\cos(\omega_0 t)\,u(t)\}$

$= \cos(\omega_0\tau)\frac{\omega_0}{s^2+{\omega_0}^2} - \sin(\omega_0\tau)\frac{s}{s^2+{\omega_0}^2}, \mathrm{Re}\{s\} > 0$

[For $\mathcal{L}\{\sin(\omega_0 t)u(t)\}$ look at Problem 4(d) solved previously.
For $\mathcal{L}\{\cos(\omega_0 t)\,u(t)\}$ look at Lecture 6-7]

# Problem 5

Find the inverse Laplace transform of the function:

$$\frac{2s+5}{s^2+5s+6}e^{-2s}$$

Consider the Laplace transform

$$F(s) = \frac{2s+5}{s^2+5s+6} = \frac{(s+3)+(s+2)}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{2s+5}{s^2+5s+6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$

$$= (e^{-2t} + e^{-3t})u(t)$$

Therefore, by using time-shifting property, we immediately see that:

$$\mathcal{L}^{-1}\left(\frac{2s+5}{s^2+5s+6}e^{-2s}\right) = \left(e^{-2(t-2)} + e^{-3(t-2)}\right)u(t-2)$$

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# Problem 6

The Laplace transform of a causal periodic signal can be found from the knowledge of the Laplace transform of its first cycle alone.

(a) If the Laplace transform of $f(t)$ shown in Fig. 6 (a) is $F(s)$, show that $G(s)$, the Laplace transform of $g(t)$ shown in Fig. 6 (b), is given by:

$$G(s) = \frac{F(s)}{1 - e^{-sT_0}}, \text{ Re}\{s\} > 0$$

$$g(t) = f(t) + f(t - T_0) + f(t - 2T_0) + \cdots$$

$$G(s) = F(s) + F(s)\, e^{-sT_0} + F(s)\, e^{-2sT_0} + \cdots$$

$$G(s) = F(s)[1 + e^{-sT_0} + e^{-2sT_0} + \cdots]$$

$$1 + r + r^2 + r^3 + \cdots = \textstyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \text{ if } |r| < 1$$

$$G(s) = F(s)[1 + e^{-sT_0} + e^{-2sT_0} + \cdots] = F(s)\,\frac{1}{1-e^{-sT_0}}, \; |e^{-sT_0}| < 1$$

Condition $|e^{-sT_0}| < 1$ is only satisfied if Re$\{s\} > 0$.

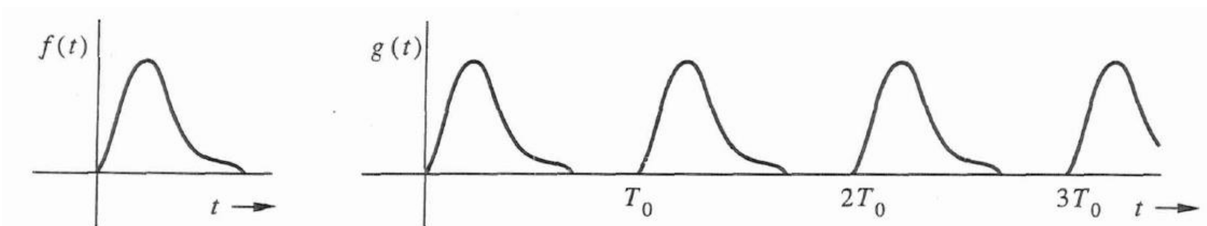


Fig. 6(a) Fig. 6(b)

# Problem 6

(b) Using the results in 6(a), find the Laplace transform of the signal $p(t)$ shown in Fig. 6 (c).

If $f(t) = u(t) - u(t-2)$, then:

$$p(t) = f(t) + f(t - T_0) + f(t - 2T_0) + \cdots$$

$$P(s) = F(s) + F(s)\, e^{-sT_0} + F(s)\, e^{-2sT_0} + \cdots$$

$$P(s) = F(s)[1 + e^{-sT_0} + e^{-2sT_0} + \cdots]$$

$$F(s) = \mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-2)\} = \frac{1}{s} - \frac{1}{s}e^{-2s}$$

$$T_0 = 8$$

$$P(s) = \frac{F(s)}{1-e^{-8s}} = \frac{1}{s} \cdot \frac{1-e^{-2s}}{1-e^{-8s}}$$

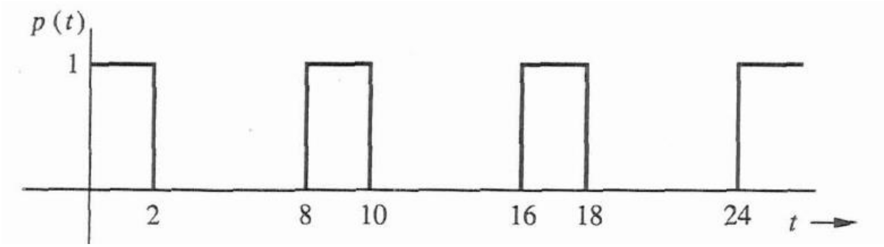


**Fig. 6(c)**

# Laplace transform pairs (1)

| No | $\boldsymbol{x}(t)$ | $\boldsymbol{X}(\boldsymbol{s})$ |
|---|---|---|
| 1 | $\delta(t)$ | $1$ |
| 2 | $u(t)$ | $\frac{1}{s}$ |
| 3 | $tu(t)$ | $\frac{1}{s^2}$ |
| 4 | $t^n u(t)$ | $\frac{n!}{s^{n+1}}$ |

# Laplace transform Pairs (2)

| No. | $x(t)$ | $X(s)$ |
|---|---|---|
| 5 | $e^{\lambda t}u(t)$ | $\frac{1}{s-\lambda}$ |
| 6 | $te^{\lambda t}u(t)$ | $\frac{1}{(s-\lambda)^2}$ |
| 7 | $t^n e^{\lambda t}u(t)$ | $\frac{n!}{(s-\lambda)^{n+1}}$ |
| 8a | $\cos bt\, u(t)$ | $\frac{s}{s^2+b^2}$ |
| 8b | $\sin bt\, u(t)$ | $\frac{b}{s^2+b^2}$ |
| 9a | $e^{-at}\cos bt\, u(t)$ | $\frac{s+a}{(s+a)^2+b^2}$ |
| 9b | $e^{-at}\sin bt\, u(t)$ | $\frac{b}{(s+a)^2+b^2}$ |

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# Laplace transform Pairs (3)

| No. | $x(t)$ | $X(s)$ |
|---|---|---|
| 10a | $re^{-at}\cos(bt+\theta)\,u(t)$ | $\dfrac{(r\cos\theta)s + (ar\cos\theta - br\sin\theta)}{s^2+2as+(a^2+b^2)}$ |
| 10b | $re^{-at}\cos(bt+\theta)\,u(t)$ | $\dfrac{0.5re^{j\theta}}{s+a-jb} + \dfrac{0.5re^{-j\theta}}{s+a+jb}$ |
| 10c | $re^{-at}\cos(bt+\theta)\,u(t)$<br>$r=\sqrt{\dfrac{A^2c+B^2-2ABa}{c-a^2}}$<br>$\theta=\tan^{-1}\left(\dfrac{Aa-B}{A\sqrt{c-a^2}}\right)$<br>$b=\sqrt{c-a^2}$ | $\dfrac{As+B}{s^2+2as+c}$ |
| 10d | $e^{-at}\left[A\cos bt + \dfrac{B-Aa}{b}\sin bt\right]u(t)$<br>$b=\sqrt{c-a^2}$ | $\dfrac{As+B}{s^2+2as+c}$ |

# Summary of Laplace transform properties (1)

| | $x(t)$ | $X(s)$ |
|---|---|---|
| Addition | $x_1(t) + x_2(t)$ | $X_1(s) + X_2(s)$ |
| Scalar multiplication | $kx(t)$ | $kX(s)$ |
| Time differentiation | $\dfrac{dx}{dt}$ | $sX(s) - x(0^-)$ |
| | $\dfrac{d^2x}{dt^2}$ | $s^2X(s) - sx(0^-) - \dot{x}(0^-)$ |
| | $\dfrac{d^3x}{dt^3}$ | $s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$ |
| | $\dfrac{d^nx}{dt^n}$ | $s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$ |
| Time integration | $\int_{0^-}^{t} x(\tau)\,d\tau$ | $\frac{1}{s}X(s)$ |
| | $\int_{-\infty}^{t} x(\tau)\,d\tau$ | $\frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$ |

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# Summary of Laplace transform properties (2)

| | $x(t)$ | $X(s)$ |
|---|---|---|
| Time shifting | $x(t-t_0)u(t-t_0)$ | $X(s)e^{-st_0} \quad t_0 \geq 0$ |
| Frequency shifting | $x(t)e^{s_0 t}$ | $X(s-s_0)$ |
| Frequency differentiation | $-tx(t)$ | $\dfrac{dX(s)}{ds}$ |
| Frequency integration | $\dfrac{x(t)}{t}$ | $\displaystyle\int_s^\infty X(z)\,dz$ |
| Scaling | $x(at),\ a \geq 0$ | $\dfrac{1}{a}X\left(\dfrac{s}{a}\right)$ |
| Time convolution | $x_1(t) * x_2(t)$ | $X_1(s)X_2(s)$ |
| Frequency convolution | $x_1(t)x_2(t)$ | $\dfrac{1}{2\pi j}X_1(s) * X_2(s)$ |
| Initial value | $x(0^+)$ | $\lim_{s\to\infty} sX(s) \quad (n > m)$ |
| Final value | $x(\infty)$ | $\lim_{s\to 0} sX(s) \quad$ [poles of $sX(s)$ in LHP] |