Imperial College London

# Signals and Systems

## Tutorial Sheet 1

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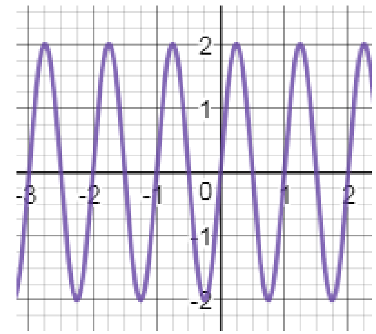
# Problem 1

Sketch each of the following continuous-time signals, specify if the signal is periodic/non-periodic, odd/even. If the signal is periodic specify its period.

(i) $x(t) = 2\sin(2\pi t)$

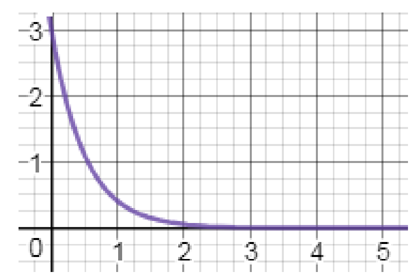
**Periodic** with period 1. **Odd** because $\sin(-t) = -\sin(t)$.




(ii) $x(t) = \begin{cases} 3e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$

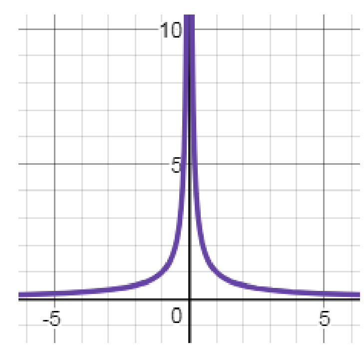
This is a **causal** system and therefore, it is **aperiodic**. A periodic system exists over the entire range of time from $-\infty$ to $\infty$.

**Neither odd nor even**. An odd or even signal must have non-zero values within both the positive and the negative range.




(iii) $x(t) = \frac{1}{|t|}$

**Aperiodic**. **Even** because $x(-t) = x(t)$.



# Problem 1 cont.

- For a signal of the form $x(t) = \sin(a\pi t) + \sin(b\pi t)$ to be periodic, the numbers $a$ and $b$ have to be **commensurable**.

  In mathematics, two non-zero real numbers $a$ and $b$ are said to be commensurable if their ratio $a/b$ is a **rational number**; otherwise $a$ and $b$ are called **incommensurable**. (Recall that a rational number is one that is equivalent to the ratio of two integers.)

  **Proof**

  We test the periodicity as follows:

  $x(t+T) = \sin\big(a\pi(t+T)\big) + \sin\big(b\pi(t+T)\big) = \sin(a\pi t)\cos(a\pi T) + \cos(a\pi t)\sin(a\pi T) + \sin(b\pi t)\cos(b\pi T) + \cos(b\pi t)\sin(b\pi T)$

  $x(t) = x(t+T)$ if $a\pi T = 2m\pi$ and $b\pi T = 2n\pi$, $m, n$ integers. Therefore,

  $$\frac{a}{b} = \frac{m}{n}$$

  Hence, the numbers $a$ and $b$ have to be commensurable.

# Problem 1 cont.

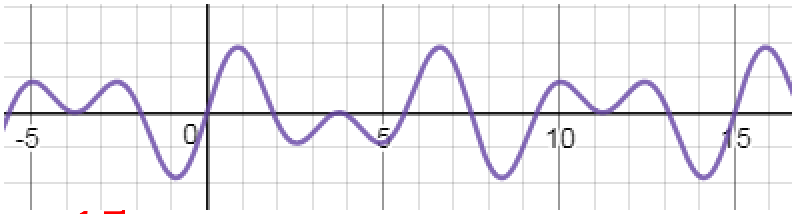
Based on the analysis of the previous slide we have:

(iv) $x(t) = \sin\left(\frac{2}{5}\pi t\right) + \sin\left(\frac{2}{3}\pi t\right), a = \frac{2}{5}, b = \frac{2}{3}, \frac{a}{b} = \frac{3}{5} = \frac{m}{n}.$

Hence, the numbers $a$ and $b$ are commensurable and the signal is **periodic**.

$$a\pi T = 2m\pi \Rightarrow T = \frac{2m}{a} = 5m$$

$$b\pi T = 2n\pi \Rightarrow T = \frac{2n}{b} = 3n.$$



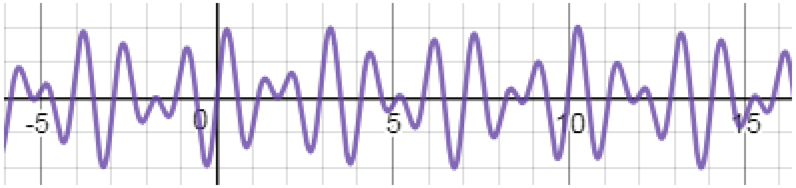

Therefore $5m = 3n \Rightarrow m = 3, n = 5,$ $\mathbf{T = 15}$.

Furthermore, $x(t)$ is **odd**.

(v) $x(t) = \sin(2\pi t) + \sin(\sqrt{2}\pi t), a = 2, b = \sqrt{2}, \frac{a}{b} = \sqrt{2} \neq \frac{m}{n}.$

Hence, the numbers $a$ and $b$ are not commensurable and the signal is **aperiodic** (although it doesn't look like in the figure!!!).

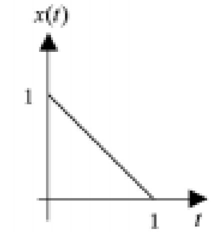
Furthermore, $x(t)$ is **odd**.



# Problem 2

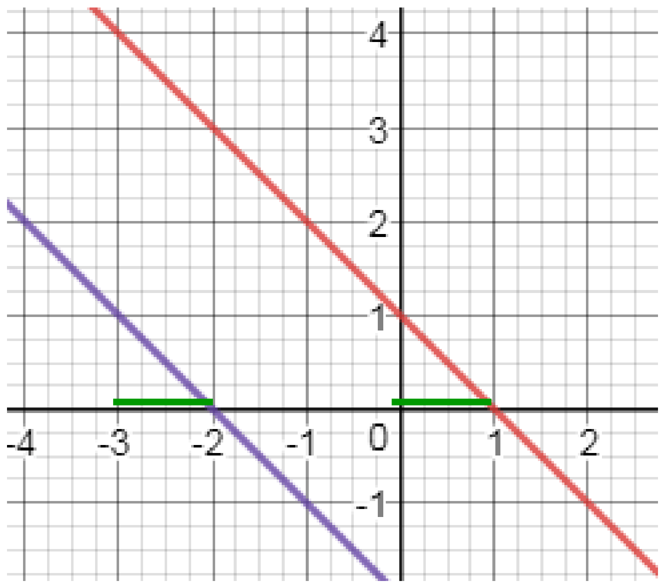
Sketch the signal:

$$x(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$




Now sketch each of the following and describe how each of the signals can be derived from the original signal $x(t)$.

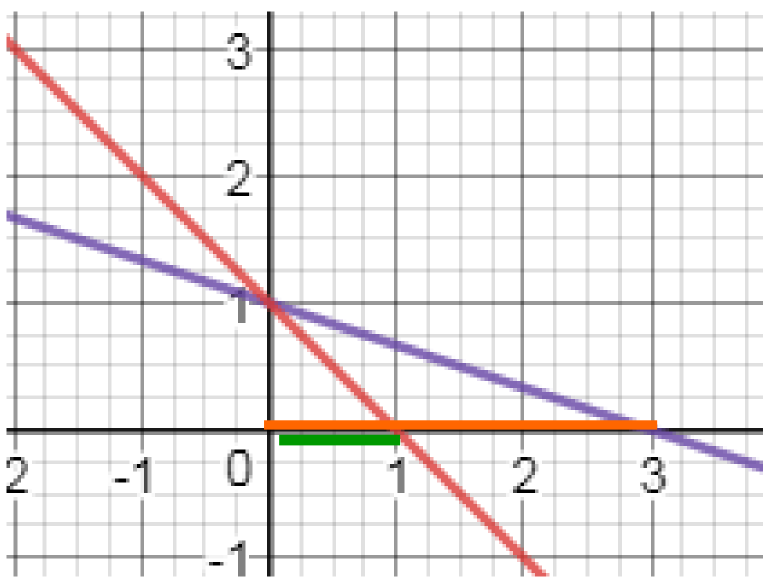
(i) $x(t+3) = \begin{cases} 1 - (t+3) = -t - 2 & 0 \le t + 3 \le 1 \Rightarrow -3 \le t \le -2 \\ 0 & \text{otherwise} \end{cases}$



# Problem 2 cont.

$$x(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) $$x(t/3) = \begin{cases} 1 - t/3 & 0 \le t/3 \le 1 \Rightarrow 0 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$



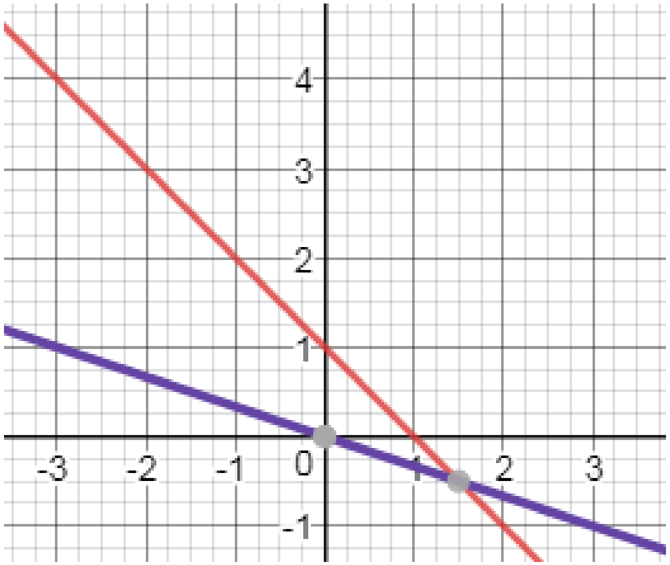
# Problem 2 cont.

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(iii)

$$x\left(\frac{t}{3} + 1\right) =$$

$$\begin{cases} 1 - \left(\frac{t}{3} + 1\right) = -\frac{t}{3} & 0 \leq \frac{t}{3} + 1 \leq 1 \Rightarrow -1 \leq \frac{t}{3} \leq 0 \Rightarrow -3 \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



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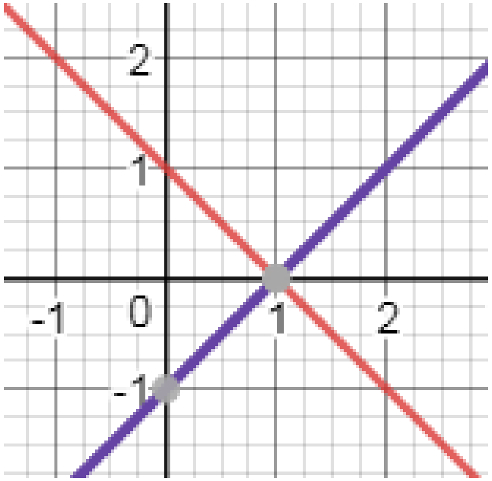
# Problem 2 cont.

$$x(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(iv)

$$x(-t+2) =$$

$$\begin{cases} 1 - (-t+2) = t - 1 & 0 \le -t + 2 \le 1 \Rightarrow -1 \le t - 2 \le 0 \Rightarrow 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$



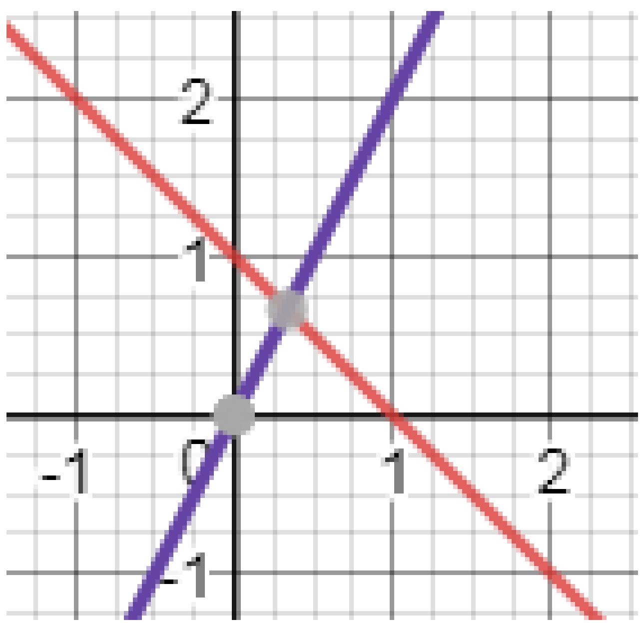
# Problem 2 cont.

$$x(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(v)

$$x(-2t + 1) =$$

$$\begin{cases} 1 - (-2t + 1) = 2t & 0 \le -2t + 1 \le 1 \Rightarrow -1 \le -2t \le 0 \Rightarrow 0 \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



# Problem 3

Sketch each of the following discrete-time signals, specify if the signal is periodic/non-periodic, odd/even. If the signal is periodic specify its period.

(i) $x[n] = \cos(n\pi)$

**Periodic** and **even**. $\cos(n+T)\pi = \cos(n\pi)\cos(T\pi) - \sin(n\pi)\sin(T\pi)$

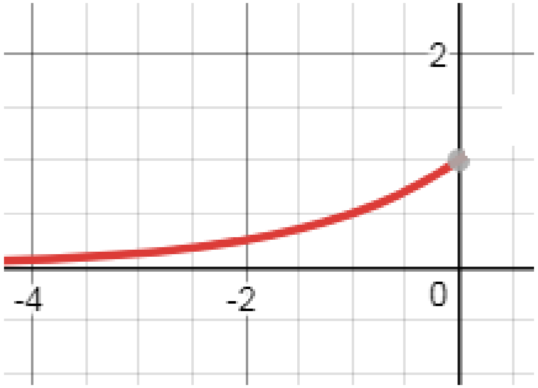
$\cos(T\pi) = 1, \sin(T\pi) = 0 \Rightarrow T = 2$.

Signal form is obvious.

(ii) $x[n] = \begin{cases} 0.5^{-n} & n \leq 0 \\ 0 & n > 0 \end{cases}$

**Aperiodic**. **Neither odd nor even**.

The continuous version is depicted on the right.



# Problem 3 cont.

(iii) What is the maximum possible frequency of the discrete exponential $e^{j\omega_0 n}$? Compare this result with the case $e^{j\omega_0 t}$.

**This question will be solved later, when we deal with the Fourier transform decompositions for discrete signals.**

# Problem 4

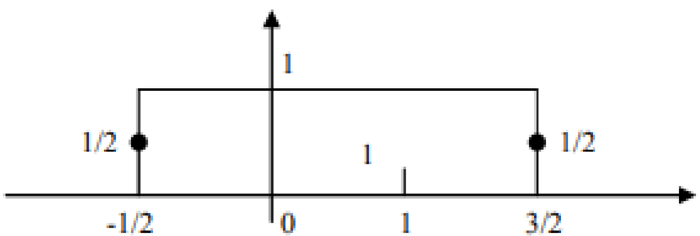
Consider the rectangular function:

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \Rightarrow -\frac{1}{2} < t < \frac{1}{2} \\ 1/2 & |t| = \frac{1}{2} \Rightarrow t = -\frac{1}{2}, \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(i) Sketch $x(t) = \sum_{k=0}^{1} \Pi(t-k)$

$$\Pi(t-1) = \begin{cases} 1 & |t-1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < t-1 < \frac{1}{2} \Rightarrow \frac{1}{2} < t < \frac{3}{2} \\ 1/2 & t = \frac{1}{2}, \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \sum_{k=0}^{1} \Pi(t-k) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{3}{2} \\ 1/2 & t = -\frac{1}{2}, \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$




(ii) Sketch $x(t) = \sum_{-\infty}^{\infty} \Pi(t-k)$. It is very easy to spot that $x(t) = 1$.

# Problem 5

Consider a discrete-time signal $x[n]$, fed as input into a system. The system produces the discrete-time output $y[n]$ such that

$$y[n] = \begin{cases} x[n] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

(i) Is the system described above memoryless? Explain

It is **memoryless** since the output at time instant $n$ depends on the input only at time instant $n$ and not past or future time instants.

(ii) Is the system described above causal? Explain.

It is **causal** since the output at time instant $n$ depends on the input only at time instant $n$ and not future time instants.

(iii) Are causal systems in general memoryless? Explain.

**No**. If the output at time instant $n$ depends on the input at time instant $n$ **and** past time instants the system is causal but not memoryless.

# Problem 5 cont.

(iv) Is the system linear and time-invariant? Explain.

The output can be written in a compact form as a function of the input as:

$$y[n] = \frac{x[n] + (-1)^n x[n]}{2}$$

We see that if the input signal $x_1[n]$ produces an output signal $y_1[n]$ and the input signal $x_2[n]$ produces an output signal $y_2[n]$ then the input signal $a_1 x_1[n] + a_2 x_2[n]$ produces the output

$$y_3[n] = \frac{(a_1 x_1[n] + a_2 x_2[n]) + (-1)^n (a_1 x_1[n] + a_2 x_2[n])}{2} = a_1 y_1[n] + a_2 y_2[n].$$

Therefore, the system is **linear**.

However, if the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n-n_o]$ produces the output $y_1[n] = \frac{x[n-n_o] + (-1)^n x[n-n_o]}{2}$.

We see that $y[n-n_o] = \frac{x[n-n_o] + (-1)^{n-n_o} x[n-n_o]}{2} \neq y_1[n]$

Therefore, the system is **time-varying**.

# Problem 6

State with a brief explanation if the following systems are linear/non-linear, causal/non-causal, time invariant/time-varying. $x[n]$, $y[n]$ are input/output.

(i) $y[n] = x[n] - x[n-1]$

We see that if the input signal $x_1[n]$ produces an output signal $y_1[n]$ and the input signal $x_2[n]$ produces an output signal $y_2[n]$ then the input signal $a_1 x_1[n] + a_2 x_2[n]$ produces the output

$y_3[n] =$

$a_1 x_1[n] + a_2 x_2[n] - a_1 x_1[n-1] - a_2 x_2[n-1] = a_1 y_1[n] + a_2 y_2[n]$

Therefore, the system is **linear**.

If the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n - n_o]$ produces the output $y_1[n] = x[n - n_o] - x[n - 1 - n_o]$

We see that $y[n - n_o] = y_1[n]$

Therefore, the system is **time invariant**.

The system is **causal** since the output does not depend on future inputs.

# Problem 6 cont.

(ii) $y[n] = \text{sgn}(x[n])$

We see that if the input signal $x_1[n]$ produces an output signal $y_1[n]$ and the input signal $x_2[n]$ produces an output signal $y_2[n]$ then the input signal $a_1x_1[n] + a_2x_2[n]$ produces the output

$$y_3[n] = \text{sgn}(a_1x_1[n] + a_2x_2[n]) \neq a_1y_1[n] + a_2y_2[n]$$

Therefore, the system is **non-linear**.

However, if the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n - n_o]$ produces the output $y_1[n] = \text{sgn}(x[n - n_o])$.

We see that $y[n - n_o] = \text{sgn}(x[n - n_o]) = y_1[n]$.

Therefore, the system is **time invariant**.

The system is **causal** since the output does not depend on future inputs.

# Problem 6 cont.

(iii) $y[n] = n^2 x[n+2]$

The input $x_1[n]$ produces an output $y_1[n]$.

The input $x_2[n]$ produces an output $y_2[n]$.

The input $a_1 x_1[n] + a_2 x_2[n]$ produces the output

$$y_3[n] = n^2(a_1 x_1[n+2] + a_2 x_2[n+2]) = a_1 y_1[n] + a_2 y_2[n]$$

Therefore, the system is **linear**.

However, if the input signal $x[n]$ produces an output signal $y[n]$ then the input signal $x[n-n_o]$ produces the output $y_1[n] = n^2 x[n-n_o+2]$.

We see that $y[n-n_o] = (n-n_o)^2 x[n-n_o+2] \neq y_1[n]$

Therefore, the system is **time-varying**.

The system is **non-causal** since if $n > 0$ then $n+2 > n$ which shows that the output requires future values of the input in order to be calculated.

# Problem 7

State with a brief explanation if the following systems are linear/non-linear, causal/non-causal, time invariant/time-varying. $x(t)$ is the input and $y(t)$ is the output.

(i) $y(t) = x(t)\cos(2\pi f_0 t + \phi)$

The input $x_1(t)$ produces the output $y_1(t) = x_1(t)\cos(2\pi f_0 t + \phi)$.
The input $x_2(t)$ produces the output $y_2(t) = x_2(t)\cos(2\pi f_0 t + \phi)$.
The linear combination $a_1 x_1(t) + a_2 x_2(t)$ produces the output
$y_3(t) = (a_1 x_1(t) + a_2 x_2(t))\cos(2\pi f_0 t + \phi) = a_1 y_1(t) + a_2 y_2(t)$
Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = x(t - t_o)\cos(2\pi f_0 t + \phi)$
We see that $y(t - t_o) = x(t - t_o)\cos(2\pi f_0(t - t_o) + \phi) \neq y_1(t)$
Therefore, the system is **time-varying**.

The system is **causal** since the output does not depend on future inputs.

# Problem 7 cont.

(ii) $y(t) = Aos(2\pi f_0 t + x(t))$

The input $x_1(t)$ produces the output $y_1(t) = A\cos\left(2\pi f_0 t + x_1(t)\right)$.

The input $x_2(t)$ produces the output $y_2(t) = A\cos\left(2\pi f_0 t + x_2(t)\right)$.

The linear combination $a_1 x_1(t) + a_2 x_2(t)$ produces the output

$y_3(t) = A\cos\left(2\pi f_0 t + a_1 x_1(t) + a_2 x_2(t)\right) \neq a_1\ y_1(t) + a_2 y_2(t)$

Therefore, the system is **non-linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = A\cos(2\pi f_0 t + x(t - t_o))$.

We see that $y(t - t_o) = A\cos(2\pi f_0 (t - t_o) + x(t - t_o)) \neq y_1(t)$

Therefore, the system is **time-varying**.

The system is **causal** since the output does not depend on future inputs.

# Problem 7 cont.

(iii) $y(t) = \int_{-\infty}^{t} x(\delta)d\delta$

The input $x_1(t)$ produces the output $y_1(t) = \int_{-\infty}^{t} x_1(\delta)d\delta$.

The input $x_2(t)$ produces the output $y_2(t) = \int_{-\infty}^{t} x_2(\delta)d\delta$ .

The linear combination $a_1x_1(t) + a_2x_2(t)$ produces the output

$$y_3(t) = \int_{-\infty}^{t} (a_1x_1(\delta) + a_2x_2(\delta))d\delta = a_1y_1(t) + a_2y_2(t)$$

Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = \int_{-\infty}^{t} x_2(\delta - t_o)d\delta.$

Replace $\delta - t_o = \tau \Rightarrow y_1(t) = \int_{-\infty}^{t-t_o} x_2(\tau)d\tau$ (observe that $d\delta = d\tau$ and if $\delta = t$ then $\tau = t - t_o$).

We see that $y(t - t_o) = \int_{-\infty}^{t-t_o} x(\delta)d\delta = y_1(t)$

Therefore, the system is **time-invariant**.

The system is **causal** since the output does not depend on future inputs.

# Problem 7 cont.

(iv) $y(t) = x(2t)$

The input $x_1(t)$ produces the output $y_1(t) = x_1(2t)$.
The input $x_2(t)$ produces the output $y_2(t) = x_2(2t)$.
The linear combination $a_1x_1(t) + a_2x_2(t)$ produces the output

$$y_3(t) = a_1x_1(2t) + a_2x_2(2t) = a_1y_1(t) + a_2y_2(t)$$

Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = x(2t - t_o)$.
We see that $y(t - t_o) = x(2(t - t_o)) \neq y_1(t)$

Therefore, the system is **time-varying**.

The system is **non-causal** since if $t > 0$ then $2t > t$ which shows that the output requires future values of the input in order to be calculated.

# Problem 7 cont.

(iv) $y(t) = x(-t)$

The input $x_1(t)$ produces the output $y_1(t) = x_1(-t)$.
The input $x_2(t)$ produces the output $y_2(t) = x_2(-t)$.
The linear combination $a_1x_1(t) + a_2x_2(t)$ produces the output

$$y_3(t) = a_1x_1(-t) + a_2x_2(-t) = a_1y_1(t) + a_2y_2(t)$$

Therefore, the system is **linear**.

If the input $x(t)$ produces an output $y(t)$, then the input $x(t - t_o)$ produces the output $y_1(t) = x(-t - t_o)$.
We see that $y(t - t_o) = x\big(-(t - t_o)\big) = x(-t + t_o) \neq y_1(t)$.

Therefore, the system is **time-varying**.

The system is **non-causal** since if $t < 0$ then $-t > t$ which shows that the output requires future values of the input in order to be calculated.