Maths for Signals and Systems

Problem Sheet 7

Singular Value Decomposition (SVD)

Consider a matrix A of dimension $m \times n$ with $m < n$ and rank r. Recall from the lectures that $r \le m$. The matrix AA^T is square, symmetric and of dimension $m \times m$. The matrix $A^T A$ is square, symmetric and of dimension $n \times n$. The following properties hold:

- Both AA^T and $A^T A$ have rank r (the same rank as the original matrix A).
- The m eigenvalues of $A^T A$ are identical to the eigenvalues of AA^T and the rest $n-m$ eigenvalues are 0.
- The so called **singular values** of A are the square roots of the non-zero eigenvalues of AA^T (or $A^T A$).
- Matrix A has a so called **Singular Value Decomposition (SVD)** of the form $A = U\Sigma V^T$ where U is of dimension $m \times m$, Σ is of dimension $m \times n$ and V is of dimension $n \times n$. Furthermore, U contains the eigenvectors of AA^T in its columns, V contains the eigenvectors of $A^T A$ in its columns and $\Sigma_{ij} = \begin{cases} \sigma_i = \sqrt{\lambda_i} & i = j, i \le r \\ 0 & \text{otherwise} \end{cases}$ with $\lambda_i, i = 1,\ldots,r$ the non-zero eigenvalues of AA^T (or $A^T A$).
- The above comments imply that $AA^T = U\Sigma^2 U^T$ and $A^T A = V\Sigma^2 V^T$.

The above analysis is straightforward in the case of $m > n$.
To understand better the structure of Σ, in the case of a 3×4 matrix of rank 2 we have $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, whereas in the case of a 4×3 matrix of rank 2 we have $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Problems

1. Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

   Solution

   We compute AA^T. (This is the smaller of the two symmetric matrices associated with A.) We get $AA^T = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$. We next find the eigenvalues of this matrix. The characteristic polynomial is $\lambda^3 - 6\lambda^2 + 6\lambda = \lambda(\lambda^2 - 6\lambda + 6)$. This gives three eigenvalues: $\lambda_1 = 3 + \sqrt{3}$, $\lambda_2 = 3 - \sqrt{3}$ and $\lambda_3 = 0$. Note that all are positive, and that there are two nonzero eigenvalues, corresponding to the fact that A has rank 2. For the singular values of A, we now take the square roots of the eigenvalues of AA^T, so $\sigma_1 = \sqrt{3+\sqrt{3}}$ and $\sigma_2 = \sqrt{3-\sqrt{3}}$. (We don't have to mention the singular values which are zero.)

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2. Find the singular values of the matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution

We use the same approach: $BB^T = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$. This has characteristic polynomial $\lambda^2 - 10\lambda + 9$ so $\lambda_1 = 9$ and $\lambda_2 = 1$ are the eigenvalues. Hence, the singular values are 3 and 1.

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3. Find the singular values of $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and find the SVD of A.

Solution

We compute AA^T and find $AA^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. The characteristic polynomial is $-\lambda^3 + 10\lambda^2 - 16\lambda = -\lambda(\lambda^2 - 10\lambda + 16) = -\lambda(\lambda - 8)(\lambda - 2)$.

The eigenvalues of AA^T are $\lambda_1 = 8$, $\lambda_2 = 2$ and $\lambda_3 = 0$. Thus, the singular values are $\sigma_1 = 2\sqrt{2}$ and $\sigma_2 = \sqrt{2}$ (and $\sigma_3 = 0$). To give the decomposition, we consider the diagonal matrix of singular values $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Next, we find an orthonormal set of eigenvectors for AA^T.

For $\lambda_1 = 8$, we find an eigenvector $[1 \quad 2 \quad 1]^T$ - normalizing gives $u_1 = [1/\sqrt{6} \quad 2/\sqrt{6} \quad 1/\sqrt{6}]^T$. For $\lambda_2 = 2$ we find $u_2 = [-1/\sqrt{3} \quad 1/\sqrt{3} \quad -1/\sqrt{3}]^T$, and finally for $\lambda_3 = 0$ we get $u_3 = [1/\sqrt{2} \quad 0 \quad -1/\sqrt{2}]^T$. This gives the matrix:

$$U = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}.$$

Finally, we have to find an orthogonal set of eigenvectors for $A^T A = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

This can be done in two ways. We show both ways, starting with orthogonal diagonalization. We already know that the eigenvalues will be $\lambda_1 = 8$, $\lambda_2 = 2$ and $\lambda_3 = 0$. This gives eigenvectors $v_1 = [1/\sqrt{6} \quad 3/\sqrt{12} \quad 1/\sqrt{12}]^T$, $v_2 = [1/\sqrt{3} \quad 0 \quad -2/\sqrt{6}]^T$ and $v_3 = [1/\sqrt{2} \quad -1/2 \quad 1/2]^T$. Put these together to get:

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 3/\sqrt{12} & 0 & -1/2 \\ 1/\sqrt{12} & -2/\sqrt{6} & 1/2 \end{bmatrix}$$

For a quicker method, we calculate the columns of V using those of U using the formula:

$$v_i = \frac{1}{\sigma_i} A^T u_i$$

$$v_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2\sqrt{2}/\sqrt{6} \\ 6/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}.$$

We can similarly calculate the other two columns.

Either way we can now verify the formula $A = U\Sigma V^T$.

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4. Find the SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Solution

We first compute $AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $A^T A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

We see immediately that the eigenvalues of AA^T are $\lambda_1 = \lambda_2 = 2$ (and hence, the eigenvalues of $A^T A$ are 2 and 0, both with multiplicity 2). Thus, the matrix A has singular value $\sigma_1 = \sigma_2 = \sqrt{2}$.

Next, an orthonormal basis of eigenvectors of AA^T is $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (You can choose any orthonormal basis of R^2 here because AA^T is a multiple of the identity, but the one chosen makes computation easiest.) Thus, we set $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For $A^T A$, the eigenvectors which correspond to the value of 2 are obtained from the formula below:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow \begin{matrix} x + z = 2x \Rightarrow z = x \\ y + w = 2y \Rightarrow y = w \\ x + z = 2z \Rightarrow x = z \\ y + w = 2w \Rightarrow y = w \end{matrix}$$

Therefore the eigenvectors which correspond to the eigenvalue of 2 are of the form $\begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Therefore two orthonormal eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

The eigenvectors which correspond to the eigenvalue of 0 are obtained from the formula below:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x + z = 0 \Rightarrow z = -x \\ y + w = 0 \Rightarrow y = -w \end{matrix}$$

Therefore the eigenvectors which correspond to the eigenvalue of 0 are of the form $\begin{bmatrix} x \\ y \\ -x \\ -y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Therefore two orthonormal eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

Therefore, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$