Maths for Signals and Systems

Problem Sheet 6

Problems

1. Consider a matrix A with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ and eigenvectors $x_1 = [\cos\theta \quad \sin\theta]^T$ and $x_2 = [-\sin\theta \quad \cos\theta]^T$. Show that $A = A^T$, $A^2 = I$, $\det(A) = -1$, $A^{-1} = A$.

Solution

The eigenvectors of this matrix are perpendicular to each other, since:

$$x_1^T x_2 = [\cos\theta \quad \sin\theta]\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 0 .$$

Furthermore, their magnitude is 1, since:

$$\|x_i\|^2 = \cos^2\theta + \sin^2\theta = 1, i = 1,2$$

For the above reasons, we conclude that the eigenvectors of matrix A are orthogonal and therefore, A is a symmetric matrix and $A = A^T$.

Moreover, A can be diagonalised as $A = Q\Lambda Q^T$ with Q a matrix which contains the eigenvectors of A in its columns, i.e., $Q = [x_1 \quad x_2]$.

$A^2 = Q\Lambda Q^T Q\Lambda Q^T = Q\Lambda^2 Q^T$ (note that $Q^T Q = I$) and $\Lambda^2 = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, so that $A^2 = Q\Lambda^2 Q^T = QIQ^T = QQ^T = I$.

$\det(A) = \lambda_1\lambda_2 = (-1)\cdot 1 = -1$.

Finally from $A^2 = I$ we see that $A^{-1} = A$.

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2. Consider a matrix A with $A^3 = 0$. Find the eigenvalues of A. Give an example of a matrix of any size that satisfies $A^3 = 0$, with $A \neq 0$. In case that a matrix A satisfies $A^3 = 0$ and is also symmetric, prove that $A = 0$.

Solution

The eigenvalues of A^3 are λ_i^3, where λ_i are the eigenvalues of A. This can be seen from the fact that if λ_i is an eigenvalue of A with corresponding eigenvector x_i then $A^3 x_i = A^2 A x_i = A^2 \lambda_i x_i = \lambda_i A A x_i = \lambda_i A \lambda_i x_i = \lambda_i^3 x_i$. But $A^3 x_i = \lambda_i^3 x_i = 0$ and therefore, $\lambda_i^3 x_i = 0 \Rightarrow \lambda_i^3 = 0 \Rightarrow \lambda_i = 0$. An example of a matrix which is non-zero but has zero eigenvalues is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $A^n = 0, \forall n$. In case where A is symmetric, the diagonalisation of A is $A = Q\Lambda Q^T$ with Q an orthogonal matrix that contains the eigenvectors of A. $A = Q\Lambda Q^T = Q0Q^T = 0$.

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3. Show that the eigenvalues of a symmetric matrix A with real entries are real.

Solution

Suppose that λ is an eigenvalue of A that corresponds to the eigenvector x. In that case we have the following:

$Ax = \lambda x \Rightarrow (Ax)^* = (\lambda x)^* \Rightarrow A^* x^* = \lambda^* x^* \Rightarrow$

$Ax^* = \lambda^* x^*$ since A is real.

In the above we transpose both sides and we get:

$(Ax^*)^T = (\lambda^* x^*)^T \Rightarrow (Ax^*)^T = \lambda^* (x^*)^T$

Now we multiply both sides from the right with x:

$(Ax^*)^T x = \lambda^* (x^*)^T x \Rightarrow (x^*)^T A^T x = \lambda^* (x^*)^T x \Rightarrow (x^*)^T Ax = \lambda^* (x^*)^T x$, since A is symmetric.

$(x^*)^T \lambda x = \lambda^* (x^*)^T x \Rightarrow \lambda (x^*)^T x = \lambda^* (x^*)^T x \Rightarrow \lambda = \lambda^*$. Therefore, λ is real.

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4. (i) A skew-symmetric (or antisymmetric) matrix B has the property $B^T = -B$. Show that the eigenvalues of a skew-symmetric matrix B with real entries are purely imaginary.
   (ii) Show that the diagonal elements of a skew-symmetric matrix are 0.

Solution

(i) $Bx = \lambda x \Rightarrow (Bx)^* = (\lambda x)^* \Rightarrow B^* x^* = \lambda^* x^* \Rightarrow$

$Bx^* = \lambda^* x^*$ since B is real.

In the above we transpose both sides and we get:

$(Bx^*)^T = (\lambda^* x^*)^T \Rightarrow (Bx^*)^T = \lambda^* (x^*)^T$

Now we multiply both sides from the right with x:

$(Bx^*)^T x = \lambda^* (x^*)^T x \Rightarrow (x^*)^T B^T x = \lambda^* (x^*)^T x \Rightarrow -(x^*)^T Bx = \lambda^* (x^*)^T x$, since B is skew-symmetric.

$-(x^*)^T \lambda x = \lambda^* (x^*)^T x \Rightarrow -\lambda (x^*)^T x = \lambda^* (x^*)^T x \Rightarrow -\lambda = \lambda^*$. Therefore, λ is purely imaginary. The only real eigenvalue that a skew-symmetric matrix might have is the zero eigenvalue.

(ii) We know that $B^T = -B$ and therefore if b_{ij} is a random element of B then $b_{ij} = -b_{ji}$. Therefore, for the diagonal elements we have $b_{ii} = -b_{ii}$ and this gives $b_{ii} = 0$.

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5. Consider the skew-symmetric matrix

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

   (i) Show that $\|Mv\| = \|v\|$, with v any 4-dimensional column vector. What observation can you make out of this result?
   (ii) Using the trace of M, the result of 5(i) and furthermore, the result of Problem 4 above, find all four eigenvalues of M.

Solution

(i)

$$Mv=\frac{1}{\sqrt{3}}\begin{bmatrix}0 & 1 & 1 & 1\\ -1 & 0 & -1 & 1\\ -1 & 1 & 0 & -1\\ -1 & -1 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=\frac{1}{\sqrt{3}}\begin{bmatrix}y+z+w\\ -x-z+w\\ -x+y-w\\ -x-y+z\end{bmatrix}$$

$$\|Mv\|^2=\frac{1}{3}\left[(y+z+w)^2+(-x-z+w)^2+(-x+y-w)^2+(-x-y+z)^2\right]$$

$$=\frac{1}{3}\left[(y^2+z^2+w^2+2yz+2zw+2yw)+(x^2+z^2+w^2-2xw-2zw+2xz)\right]$$

$$+\frac{1}{3}\left[(x^2+y^2+w^2+2xw-2yw-2xy)+(x^2+y^2+z^2-2xz-2yz+2xy)\right]$$

$$=\frac{1}{3}\left[(y^2+z^2+w^2)+(x^2+z^2+w^2)\right]+\frac{1}{3}\left[(y^2+z^2+w^2)+(x^2+z^2+w^2)\right]$$

$$=x^2+y^2+z^2+w^2=\|v\|^2\Rightarrow\|Mv\|=\|v\|.$$

If we take an eigenvalue λ of matrix M which correspond to the eigenvector v, we have that $Mv=\lambda v\Rightarrow\|Mv\|=\|\lambda v\|$. Based on the result $\|Mv\|=\|v\|$ we have $\|v\|=\|\lambda v\|=\|\lambda\|\|v\|\Rightarrow\|\lambda\|=1$. Therefore, the eigenvalues of the given matrix have magnitude 1.

(ii) In Problem 4 we proved that the eigenvalues of a skew-symmetric matrix are purely imaginary. For this particular case we also see that they have magnitude of 1. Therefore, for matrix M we can say that the eigenvalues are i or $-i$ or 0. The matrix has four eigenvalues and they sum up to zero. The determinant of M is not zero (it is actually $\frac{1}{\left(\sqrt{3}\right)^4}9=1$) which means that the matrix is full rank and therefore, it doesn't have any zero eigenvalues. Therefore, two of them are equal to i and the rest are equal to $-i$.

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6. Consider matrices A and B shown below:

$$A=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\text{ and }B=\frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

(i) In which of these classes do they belong to? Invertible, orthogonal, projection, permutation, diagonalisable, Markov.

(ii) Which of the factorisations $LU, QR, S\Lambda S^{-1}, Q\Lambda Q^{-1}$ are possible for A and B?

Solution

(i) For A we have:

Eigenvalues are -1, 1, 1 and corresponding eigenvectors are $\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^T$, $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T$ and $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$. The eigenvectors are independent and also orthogonal. The determinant is -1.

A is invertible (its determinant is non-zero), orthogonal (its rows are orthogonal to each other), permutation (obvious), diagonalisable (any invertible matrix is diagonalisable), Markov (satisfies the Markov properties – rows/columns have positive elements which sum up to 1). It is not a projection matrix because it doesn't satisfy the property $A^2=A$.

For B we have:

Eigenvalues are 1, 0, 0 and corresponding eigenvectors are $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$, $\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^T$ and $\begin{bmatrix}-1 & 1 & 0\end{bmatrix}^T$. The eigenvectors are independent and also orthogonal. The determinant is 0.

B is projection ($B^2 = B$ and is symmetric), diagonalizable (it has a set of independent eigenvectors), Markov (satisfies the Markov properties – rows/columns have positive elements which sum up to 1). It is not orthogonal or permutation.

(ii) For a 3×3 matrix the LU decomposition looks like:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = LU$$

For the given matrix A we have $a_{11} = 0$ and therefore, at least one of l_{11} and u_{11} has to be zero. In that case either L or U is singular. This is not possible since A is not singular. Therefore, A doesn't have an LU decomposition. In order for A to have an LU decomposition, we must reorder the rows of A, i.e., A must be multiplied from the left with a permutation matrix P, and in that case we have $PA = LU$.

A has a QR decomposition as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the given matrix B we find the LU decomposition using elimination, as follows:

$$B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$ and therefore, B does have an LU decomposition.

$$B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.577 & 0.816 & 0 \\ 0.577 & -0.408 & 0.707 \\ 0.577 & -0.408 & -0.707 \end{bmatrix} \begin{bmatrix} 0.577 & 0.577 & 0.577 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ where the matrix $\begin{bmatrix} 0.577 & 0.816 & 0 \\ 0.577 & -0.408 & 0.707 \\ 0.577 & -0.408 & -0.707 \end{bmatrix}$ is orthogonal, therefore B does have a QR decomposition.

Both matrices have $S\Lambda S^{-1}$ decomposition since they have a set of independent eigenvectors.

Both matrices have $Q\Lambda Q^{-1}$ decomposition since, due to their symmetry, we can choose a set of independent and also orthogonal eigenvectors.
