Maths for Signals and Systems

Problem Sheet 5

Problems

1. The Lucas numbers are similar to the Fibonacci numbers but the initial conditions are $L_1 = 1$ and $L_2 = 3$. The relationship $L_{k+2} = L_{k+1} + L_k$ holds. Find L_{100}.

Solution

We define u_k to be the 2-dimensional column vector $u_k = \begin{bmatrix} L_{k+1} \\ L_k \end{bmatrix}$. In that case $u_{k+1} = \begin{bmatrix} L_{k+2} \\ L_{k+1} \end{bmatrix}$.
We also create the "fake" equation $L_{k+1} = L_{k+1}$. From the two equations:

$$L_{k+2} = L_{k+1} + L_k$$
$$L_{k+1} = L_{k+1}$$

we can formulate the matrix form:

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

The eigenvalues of the matrix are obtained from $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda + \lambda^2 - 1 = 0 \Rightarrow \lambda^2 = \lambda + 1$ and they are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. The eigenvectors are obtained from:
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x_i = \lambda_i x_i \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} = \lambda_i \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} \Rightarrow x_{i1} + x_{i2} = \lambda_i x_{i1}$ and $x_{i1} = \lambda_i x_{i2}$, $i = 1,2$. By choosing $x_{i2} = 1$, we have $x_i = \begin{bmatrix} \lambda_i \\ 1 \end{bmatrix}$.

We write $u_1 = \begin{bmatrix} L_2 \\ L_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We assume that $u_1 = c_1 x_1 + c_2 x_2 \Rightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.
This gives us

$$3 = c_1 \frac{1+\sqrt{5}}{2} + (1-c_1)\frac{1-\sqrt{5}}{2} = c_1\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) + \frac{1-\sqrt{5}}{2} \Rightarrow \sqrt{5}c_1 = \frac{5+\sqrt{5}}{2} \Rightarrow c_1 = \frac{1+\sqrt{5}}{2} = \lambda_1$$

$c_2 = 1 - c_1 = 1 - \frac{1+\sqrt{5}}{2} = \lambda_2$.
We have $A = S\Lambda S^{-1}$ and $A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$ and in general $A^n = S\Lambda^n S^{-1}$. We see that $u_2 = Au_1$, $u_3 = Au_2 = AAu_1 = A^2 u_1$ and generally $u_{k+1} = A^k u_1$. Therefore, we have $u_{100} = A^{99} u_1 = S\Lambda^{99} S^{-1} u_1$.

$$S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix},\quad S^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \text{ and } S^{-1}u_1 = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$\Lambda^{99}S^{-1}u_1 = \Lambda^{99}\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} = \begin{bmatrix}\lambda_1^{100} \\ \lambda_2^{100}\end{bmatrix}$ and

$u_{100} = \begin{bmatrix}L_{101} \\ L_{100}\end{bmatrix} = A^{99}u_1 = S\Lambda^{99}S^{-1}u_1 = \begin{bmatrix}\lambda_1 & \lambda_2 \\ 1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^{100} \\ \lambda_2^{100}\end{bmatrix} = \begin{bmatrix}\lambda_1^{101} + \lambda_2^{101} \\ \lambda_1^{100} + \lambda_2^{100}\end{bmatrix}$ and therefore

$L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

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2. Find the inverse, the eigenvalues and the determinant of A

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$$

Solution

$A = 5I - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 5I - B$ where B is a matrix of size 4×4 with all its elements equal to 1.

We observe that A is the sum of two symmetric matrices. Matrix $5I$ has eigenvalues $5,5,5,5$ and matrix B is singular with rank 1. This is quite obvious since its rows are the same. Therefore, matrix B has an eigenvalue $\lambda = 0$ repeated 3 times. The 4^{th} eigenvalue of B is obtained from the trace of B and is equal to $\lambda = 4$. Matrices $5I$ and B are symmetric and therefore, we can always find a set of orthonormal eigenvectors for them. If we consider the matrix B its eigenvector that corresponds to $\lambda = 4$ is $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$. The rest of the eigenvectors are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x + y + z + w = 0$$

A nice choice for the eigenvector set is the set of the rows of the so called Hadamard matrix.

$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

These eigenvectors can also be chosen to be the eigenvectors of the matrix $5I$. Since we can assume that $5I$ and B share the same set of eigenvectors we can assume that the eigenvalues of A are equal to the eigenvalues of $5I$ minus the eigenvalues of B. Therefore, the eigenvalues of A are $1,5,5,5$. Its eigenvectors are the same as the eigenvectors of $5I$ and B, i.e., the rows of the Hadamard matrix shown above.

The determinant of A is $1 \cdot 5 \cdot 5 \cdot 5 = 125$.

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3. (i) Carry out the eigenvalue decomposition of the matrices $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}$.

(ii) Carry out the eigenvalue decomposition of A^3, B^3 and A^{-1}.

Solution

(i) The eigenvalues of $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$ are found from $\begin{vmatrix}1-\lambda & 2\\ 0 & 3-\lambda\end{vmatrix}=0\Rightarrow(1-\lambda)(3-\lambda)=0$ and they are $\lambda=1,\lambda=3$. The eigenvector that correspond to $\lambda=1$ is found from $\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}$. We have $3y=y\Rightarrow y=0$ and therefore the eigenvector is of the form $\begin{bmatrix}x\\ 0\end{bmatrix}$ with x a non-zero integer. The eigenvector that correspond to $\lambda=3$ is found from $\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=3\begin{bmatrix}x\\ y\end{bmatrix}$. We have $2x=2y\Rightarrow x=y$ and therefore the eigenvector is of the form $\begin{bmatrix}x\\ x\end{bmatrix}$ with x a non-zero integer. Therefore $S=\begin{bmatrix}x & x\\ 0 & x\end{bmatrix}$ and $S^{-1}=\frac{1}{x^2}\begin{bmatrix}x & -x\\ 0 & x\end{bmatrix}$. The eigenvalue decomposition of $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$ is $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}=\frac{1}{x^2}\begin{bmatrix}x & x\\ 0 & x\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}x & -x\\ 0 & x\end{bmatrix}$.

The eigenvalues of $B=\begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}$ are found from $\begin{vmatrix}1-\lambda & 1\\ 3 & 3-\lambda\end{vmatrix}=0\Rightarrow$ $(1-\lambda)(3-\lambda)-3=0\Rightarrow\lambda^2-4\lambda=0\Rightarrow\lambda=0,\lambda=4$. The eigenvector that correspond to $\lambda=0$ is found from $\begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. We have $x=-y$ and therefore the eigenvector is of the form $\begin{bmatrix}x\\ -x\end{bmatrix}$ with x a non-zero integer. The eigenvector that correspond to $\lambda=4$ is found from $\begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=4\begin{bmatrix}x\\ y\end{bmatrix}$. We have $y=3x$ and therefore the eigenvector is of the form $\begin{bmatrix}x\\ 3x\end{bmatrix}$ with x a non-zero integer. Therefore $S=\begin{bmatrix}x & x\\ -x & 3x\end{bmatrix}$ and $S^{-1}=\frac{1}{4x^2}\begin{bmatrix}3x & -x\\ x & x\end{bmatrix}$. The eigenvalue decomposition of $B=\begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}$ is $B=\begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}=\frac{1}{4x^2}\begin{bmatrix}x & x\\ -x & 3x\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}3x & -x\\ x & x\end{bmatrix}$.

(ii) $A^2=\frac{1}{x^2}\begin{bmatrix}x & x\\ 0 & x\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}x & -x\\ 0 & x\end{bmatrix}$ and $A^{-1}=\frac{1}{x^2}\begin{bmatrix}x & x\\ 0 & x\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}x & -x\\ 0 & x\end{bmatrix}$.

$B^2=\frac{1}{4x^2}\begin{bmatrix}x & x\\ -x & 3x\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 16\end{bmatrix}\begin{bmatrix}3x & -x\\ x & x\end{bmatrix}$.

In this question we used the fact that the eigenvalues of the inverse of a matrix are the inverses of the eigenvalues of the original matrix.

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4. (i) Suppose that $A=S\Lambda S^{-1}$. What is the eigenvalue matrix of $A+2I$?

(ii) What is the eigenvector matrix of $A+2I$?
(iii) Carry out the eigenvector decomposition of $A+2I$.

Solution

(i) The eigenvectors of $2I$ can be chosen to be the eigenvectors which form the matrix S in $A=S\Lambda S^{-1}$. Since we can assume that A and $2I$ share the same set of eigenvectors we can assume that the eigenvalues of $A+2I$ are equal to the eigenvalues of A plus the eigenvalues of $2I$. Therefore, the eigenvalue matrix of $A+2I$ is $\Lambda+2I$.
(ii) It is obvious from the above analysis that the eigenvector matrix of $2I$ is S .
(iii) $A+2I=S(\Lambda+2I)S^{-1}=S\Lambda S^{-1}+S2IS^{-1}=A+2I$

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5. Consider the matrix $A=I_{n\times n}+B_{n\times n}$ where $B_{n\times n}$ is a matrix with all its elements equal to 1, of size $n\times n$. It is given that $A^{-1}=I_{n\times n}+cB_{n\times n}$. Find c .

Solution

$AA^{-1}=I_{n\times n}=(I_{n\times n}+B_{n\times n})(I_{n\times n}+cB_{n\times n})=I_{n\times n}I_{n\times n}+cI_{n\times n}B_{n\times n}+B_{n\times n}I_{n\times n}+cB_{n\times n}B_{n\times n}$
$=I_{n\times n}+cB_{n\times n}+B_{n\times n}+cB_{n\times n}{}^2\Rightarrow cB_{n\times n}+B_{n\times n}+cB_{n\times n}{}^2=0$
We can easily see that $B_{n\times n}{}^2=nB_{n\times n}$.
Therefore, $cB_{n\times n}+B_{n\times n}+cnB_{n\times n}=0\Rightarrow c+1+cn=0\Rightarrow c=\dfrac{-1}{n+1}$.
