Maths for Signals and Systems

Problem Sheet 4

Problems

1. Consider a matrix A of size 3×3. Using the properties of determinants, find the determinants of three matrices $M_i, i=1,2,3$ which are obtained from A through the following operations:
   (i) M_1 is obtained by multiplying each element a_{ij} of A with $(-1)^{i+j}$.
   (ii) M_2 is obtained when rows 1,2,3 of A are subtracted from rows 2,3,1.
   (iii) M_3 is obtained when rows 1,2,3 of A are added to rows 2,3,1.

Solution

(i) The matrix M_1 is given as follows:

$$M_1 = \begin{bmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ a_{31} & -a_{32} & a_{33} \end{bmatrix}$$. This can be written as

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $\det(M_1) = (-1)\det(A)(-1) = \det(A)$.

(ii) $$M_2 = \begin{bmatrix} a_{11}-a_{31} & a_{12}-a_{32} & a_{13}-a_{33} \\ a_{21}-a_{11} & a_{22}-a_{12} & a_{23}-a_{13} \\ a_{31}-a_{21} & a_{32}-a_{22} & a_{33}-a_{23} \end{bmatrix}$$. The rows of M_2 add up to 0 and therefore, they are dependent. The determinant of M_2 is 0.

(iii) $$M_3 = \begin{bmatrix} \text{row}1+\text{row}3 \\ \text{row}2+\text{row}1 \\ \text{row}3+\text{row}2 \end{bmatrix}$$

$$\det(M_3) = \begin{vmatrix} \text{row}1+\text{row}3 \\ \text{row}2+\text{row}1 \\ \text{row}3+\text{row}2 \end{vmatrix} = \begin{vmatrix} \text{row}1 \\ \text{row}2+\text{row}1 \\ \text{row}3+\text{row}2 \end{vmatrix} + \begin{vmatrix} \text{row}3 \\ \text{row}2+\text{row}1 \\ \text{row}3+\text{row}2 \end{vmatrix}$$

$$= \begin{vmatrix} \text{row}1 \\ \text{row}2 \\ \text{row}3+\text{row}2 \end{vmatrix} + \begin{vmatrix} \text{row}1 \\ \text{row}1 \\ \text{row}3+\text{row}2 \end{vmatrix} + \begin{vmatrix} \text{row}3 \\ \text{row}2 \\ \text{row}3+\text{row}2 \end{vmatrix} + \begin{vmatrix} \text{row}3 \\ \text{row}1 \\ \text{row}3+\text{row}2 \end{vmatrix}$$

$$= \begin{vmatrix} \text{row}1 \\ \text{row}2 \\ \text{row}3 \end{vmatrix} + \begin{vmatrix} \text{row}1 \\ \text{row}2 \\ \text{row}2 \end{vmatrix} + \begin{vmatrix} \text{row}1 \\ \text{row}1 \\ \text{row}3 \end{vmatrix} + \begin{vmatrix} \text{row}1 \\ \text{row}1 \\ \text{row}2 \end{vmatrix} + \begin{vmatrix} \text{row}3 \\ \text{row}2 \\ \text{row}3 \end{vmatrix} + \begin{vmatrix} \text{row}3 \\ \text{row}2 \\ \text{row}2 \end{vmatrix} + \begin{vmatrix} \text{row}3 \\ \text{row}1 \\ \text{row}3 \end{vmatrix} + \begin{vmatrix} \text{row}3 \\ \text{row}1 \\ \text{row}2 \end{vmatrix} =$$

$$= \det(A) + 0 + 0 + 0 + 0 + 0 + 0 + (-1)(-1)\det(A) = 2\det(A).$$

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2. Find the determinant of the following matrix and investigate whether the matrix is singular for certain values of the parameter a.

$$A = \begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix}$$

Solution

We subtract row 3 from row 1. In that case the matrix becomes:

$$\begin{bmatrix} -a & 0 & a \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix}$$

Then we subtract row 3 from row 2. The matrix now becomes:

$$\begin{bmatrix} -a & 0 & a \\ 0 & -a & a \\ 1 & 1 & 1-a \end{bmatrix}$$

We obtain the determinant using the cofactors of the first row. Note that when we replace a row with a linear combination of rows the determinant doesn't change. In that case we have $\det(A) = -a[-a(1-a)-a] + a(0+a) = -a(-a+a^2-a-a) = -a(a^2-3a) = -a^2(a-3)$.

The determinant is zero in the following two cases:

- $a = 0$. In that case the original matrix is the "all ones" matrix which is obviously singular.
- $a = 3$. In that case the original matrix becomes $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$. In that case the rows or columns add up to 0, and therefore, the matrix is singular.

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3. (i) If the entries in every row of a matrix A add to zero, prove that $\det(A) = 0$ by commenting on the solutions of the system $Ax = 0$.
(ii) If the entries of every row of a matrix add up to 1, show that $\det(A-I) = 0$. Does this mean that $\det(A) = \det(I) = 1$?

Solution

(i) If the entries in every row add up to zero the "all ones" column vector is in the null space. Therefore, the null space is a non-zero subspace and therefore, the original matrix is singular.
(ii) If the entries of every row of the matrix add up to 1, the entries in every row of $A-I$ add up to zero. Based on the comments of (i) we immediately see that $\det(A-I) = 0$. That doesn't mean that $\det(A) = \det(I) = 1$, since in general $\det(A+B) \neq \det(A) + \det(B)$.

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4. A Hessenberg matrix is a square triangular matrix with one extra non-zero diagonal. The 2×2, 3×3 and 4×4 Hessenberg matrices are shown below:

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},\ H_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule continues for all sizes, i.e., $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

Solution

The cofactor C_{11} for H_4 is the determinant $|H_3|$. We also need the cofactor C_{12}. This is $C_{12} = -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$. If we consider matrix H_3, this can be written as $H_3 = \begin{bmatrix} 1+1 & 1+0 & 0+0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Therefore,

$$|H_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \Rightarrow C_{12} = -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -|H_3| + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \Rightarrow$$

$$C_{12} = -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -|H_3| + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -|H_3| + 1 \cdot |H_2| = -|H_3| + |H_2|$$

$$|H_4| = 2C_{11} + C_{12} = 2|H_3| + (-|H_3| + |H_2|) = |H_3| + |H_2|$$

We have $|H_2| = 3$, $|H_3| = 5$, $|H_4| = 8$. By considering the Fibonacci sequence we immediately see that $|H_n| = F_{n+2}$.

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5. (i) If A is the 10 by 10 "all-ones" matrix, how does the big formula for determinant give $\det(A) = 0$?

   (ii) If we multiply all $n!$ permutation matrices of size $n \times n$ is the resulting matrix's determinant +1 or -1?

   (iii) If we multiply each element a_{ij} of a matrix with the fraction $\frac{i}{j}$, how is the determinant of the matrix affected?

Solution

(i) In the big formula (please look at your notes), in case of a square matrix with even number of rows and columns, half of the products will be +1 and half of them will be -1. Therefore, the determinant will be zero. This is anyway expected, since the 10 by 10 "all-ones" matrix is singular.

(ii) In the case of 2 by 2 matrices we have two distinct permutation matrices. These are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with determinants +1 and -1 respectively. Therefore, the determinant of the product is -1.

In case of 3 by 3 matrices we have the following permutations:

$\begin{vmatrix} \text{row}1 \\ \text{row}2 \\ \text{row}3 \end{vmatrix}, \begin{vmatrix} \text{row}1 \\ \text{row}3 \\ \text{row}2 \end{vmatrix}, \begin{vmatrix} \text{row}2 \\ \text{row}1 \\ \text{row}3 \end{vmatrix}, \begin{vmatrix} \text{row}2 \\ \text{row}3 \\ \text{row}1 \end{vmatrix}, \begin{vmatrix} \text{row}3 \\ \text{row}1 \\ \text{row}2 \end{vmatrix}, \begin{vmatrix} \text{row}3 \\ \text{row}2 \\ \text{row}1 \end{vmatrix}$ with determinants $1,-1,-1,1,-1,1$.

There is an odd number of -1s since in that case there are $\frac{6!}{2} = 3$ permutations with an odd number of row exchanges. In that case the determinant of the product is $1 \times (-1) \times (-1) \times 1 \times (-1) \times 1 = -1$.

For matrices of size greater than 3 we have that the number of permutation matrices with odd number of row exchanges is $\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n}{2} = 3 \cdot 4 \cdot \ldots \cdot n, n \geq 4$. This is always even and therefore we have an even number of determinants equal to -1 and therefore, the total determinant is 1.

(iii) In the "big formula" for the estimation of the determinant we have sums of products of the form $a_{1a} a_{2b} \cdot \ldots \cdot a_{nx}$ where all rows and all columns participate once. Therefore, if we multiply each element a_{ij} of a matrix with the fraction $\frac{i}{j}$, the term $a_{1a} a_{2b} \cdot \ldots \cdot a_{nx}$ will be multiplied by all row numbers and divided by all column numbers and therefore will remain unchanged. Therefore, in that case the determinant remains unchanged.