Maths for Signals and Systems

Problem Sheet 2

Problems

1. Find the best straight line that fits the points: (0,0),(1,8),(3,8),(4,20). Find the square of the magnitude of the error vector of the approximation. This is called the **Eucledian norm** (or **second norm**) of the error vector. If the error vector is denoted with a column vector e, its Eucledian norm is the sum of the squares of the elements of the vector and is obtained by $e^T e$ which is the inner product of the vector with itself.

Solution

The required line is described by the equation $y = C + Dt$. If we assume that the given points lie on that line then we formulate the following set of equations:

$$C = 0$$
$$C + D = 8$$
$$C + 3D = 8$$
$$C + 4D = 20$$

and therefore the system that we are required to solve is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

We have $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Therefore, $A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^T b = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

We solve the system $A^T A\hat{x} = A^T b \Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \hat{x} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$ and obtain $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. In that case we have

$p = A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $e = b - p = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$. Therefore, $\|e\|^2 = 44$.

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2. Consider nine approximate measurements of some quantities $b_1, \ldots, b_9$. All measurements are zero. The measurements are taken at times $t = 1, \ldots, 9$. Furthermore, at time $t = 10$ a tenth measurement which is equal to 40 is taken. This last measurement is very different from the others and it is very likely that it is wrong. In engineering and statistics these types of measurements are called **outliers**. Find a single final "best" measurement that:
(i) Minimizes the Eucledian norm of the error vector for this approximation.

(ii) Minimizes the maximum absolute error. This is equivalent of minimizing the maximum element of the error vector. This function is called **infinity norm** and it is denoted by $E_\infty = |e_{\max}|$.

(iii) Minimizes the sum of the magnitudes of the elements of the error vector. This function is called **first norm** and it is denoted by $E_1 = |e_1| + |e_2| + \ldots + |e_{10}|$.

Solution

(i) We are looking for the best horizontal line $y = C$ that fits the 10 points (1,0), (2,0),…,(9,0),(10,40). Since our measurements are 0 and one of them 40, the horizontal line must be between levels 0 and 40.
We must solve the system:

$$\begin{aligned} C &= 0 \\ C &= 0 \\ C &= 0 \\ C &= 0 \\ C &= 0 \\ C &= 0 \\ C &= 0 \\ C &= 0 \\ C &= 0 \\ C &= 40 \end{aligned}$$

which is written in matrix form as follows:

$$AC = b \Rightarrow A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 40 \end{bmatrix}$$

The approach that minimizes the Eucledian norm of the error is the solution obtained by projecting the vector b onto the column space of A.
We have $A^T A = 10$ and $A^T b = 40$. Projection gives

$$A^T AC = A^T b \Rightarrow 10C = 40 \Rightarrow C = 4$$

(ii) The error of the approximation is the vector

$$\begin{bmatrix} -C & -C & -C & -C & -C & -C & -C & -C & -C & 40-C \end{bmatrix}^T$$

Knowing that the required straight line is $y = C, 0 \le C \le 40$ the absolute error is

$$\begin{bmatrix} C & C & C & C & C & C & C & C & C & 40-C \end{bmatrix}^T$$

In case $C \ge 40 - C \Rightarrow 2C \ge 40 \Rightarrow C \ge 20$ then the maximum error vector elements are C and since $C \ge 20$ these are minimized at $C = 20$.
In case $C \le 40 - C \Rightarrow 2C \le 40 \Rightarrow C \le 20$ then the maximum error vector element is $40 - C$ and since $C \le 20 \Rightarrow -C \ge -20 \Rightarrow 40 - C \ge 20$ this is minimized at $40 - C = 20 \Rightarrow C = 20$.

Therefore the answer is $C = 20$.

(iii) The sum of the magnitudes of the individual measurements' errors is $9C + 40 - C = 40 + 8C$. This is minimized at $C = 0$.
