Maths for Signals and Systems

Problem Sheet 2

In the formulation $Ax = 0$, where A is a matrix of size $m \times n$ and x is a column vector of size n, then 0 is a column vector with m zero elements.

Problem 1
Show that a basis for the left null space of a rectangular matrix A of size $m \times n$ consists of the last $m-r$ rows of the corresponding elimination matrix E.
Solution
The left null space is obtained by the solutions of the system $A^T x = 0$, or equivalently, $x^T A = 0^T$, where x is a column vector and therefore, x^T is a row vector.
As we discussed in detail in the lecture, the reduced row echelon form (rref) of matrix A is obtained by a sequence of operations imposed on matrix A. Each operation involves replacing a row of A with a linear combination of itself and another row of A, with the goal of making this row look "simpler" by replacing at least one element of the row with 0. This operation is equivalent with multiplying A from the left with a matrix E_{ij} of size $m \times m$. The exact form of matrix E_{ij} is given and justified in your lecture notes. We realise a number of this type of operations until A cannot be simplified further. In that case we can write

$$EA = R$$

where $E = \prod_{ij} E_{ij}$ is the product of all elimination matrices used in the procedure. We can find the matrix E automatically if instead of carrying elimination on A only, we carry elimination on the augmented matrix $[A \vdots I_{m\times m}]$ where $I_{m\times m}$ is the identity matrix of size $m \times m$. In that case after elimination the augmented matrix $[A \vdots I_{m\times m}]$ becomes:

$$E[A \vdots I_{m\times m}] = [EA \vdots EI_{m\times m}] = [R \vdots E]$$

Therefore we can easily obtain E.
If the rank of matrix A is r then the last $m-r$ rows of R are zero rows. Therefore, from the equation $EA = R$ we see that each of the last $m-r$ rows of E multiplied with A from the left gives a zero row vector. This verifies the fact that the last $m-r$ rows of E belong to the left null space, since they satisfy the relationship $x^T A = 0^T$. Due to the method that we use to construct E, it can be shown easily that E is a full rank matrix (rank is m) and therefore its last $m-r$ rows are independent. Since these rows belong to the left null space and knowing that the left null space has dimension $m-r$, we can say that the last $m-r$ rows of E form a basis of the left null space.

Problem 2
Construct a matrix with the characteristics described in (a)-(d) below or if what is required is not possible, explain why it is impossible.

(a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix}1\\1\\3\end{bmatrix}$, null space has basis $\begin{bmatrix}3\\1\\1\end{bmatrix}$.

(c) Dimension of null space = 1 + dimension of left null space.

(d) Left null space contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.

Solution

(a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$.This question refers to a matrix A of size 3×2. This can be verified from the fact that the rows have two elements and the columns have three elements. Since the column space and the row space have the same dimension (rank), the maximum rank of this matrix is 2. The row space contains the 2 given two-dimensional vectors, which are independent. Therefore, the rank of A **is** 2. If we formulate the matrix $A=\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix}$ using the given columns we observe that the first and the third rows are independent. Furthermore, the given rows can be easily written as linear combinations of the rows of $A=\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix}$. Therefore, this A satisfied the requirements of the question.

(b) The required matrix has 3 rows, since the column space has a three dimensional vector as a basis. The basis of the column space is a single vector and therefore, the dimension of the column space is 1. That means that the rank of the matrix is 1. The null space has as basis a three dimensional vector as well, and therefore, the required matrix has 3 columns. This means that the required matrix is a square matrix of size 3×3. In that case the dimensions of column (or row) space and null space should add up to three. This is not the case and therefore, it is not possible to construct the required matrix.

(c) The dimension of null space is $n-r$ and the dimension of the left null space is $m-r$. Therefore, according to the given formula $n-r=1+m-r \Rightarrow n=m+1$. Therefore any matrix where the number of columns is larger by 1 compared to number of rows satisfies the required conditions.

(d) Since left the null space contains a two dimensional vector, the required matrix has two **rows**. Since the row space contains also a two dimensional vector, the required matrix has two **columns**. Therefore the required matrix is a square matrix of size 2×2. Suppose the matrix is $A=\begin{bmatrix}a & b\\3 & 1\end{bmatrix}$. In that case $A^T x=0^T$ gives $\begin{bmatrix}a & 3\\b & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, and therefore, $a+9=0 \Rightarrow a=-9$ and $b+3=0 \Rightarrow b=-3$. This gives $A=\begin{bmatrix}-9 & -3\\3 & 1\end{bmatrix}$.

Problem 3

Explain why $v=\begin{bmatrix}1 & 0 & -1\end{bmatrix}$ cannot be a row of A and also in the null space.

Solution

The null space must consist of three dimensional vectors, therefore A must have 3 columns. Since $v=\begin{bmatrix}1 & 0 & -1\end{bmatrix}$is a row of A and a vector of the null space, the inner product of $v=\begin{bmatrix}1 & 0 & -1\end{bmatrix}$ with itself should be zero. This cannot be true for any nonzero vector since the inner product of a vector with itself is the sum of the squares of the elements of that vector.

Problem 4

(a) Verify that the special solutions to $Ax=0$ are perpendicular to the rows of A using the formula and matrices given below:

$$A=\begin{bmatrix}1&0&0\\2&1&0\\3&4&1\end{bmatrix}\begin{bmatrix}4&2&0&1\\0&0&1&3\\0&0&0&0\end{bmatrix}=E^{-1}R$$

(b) Find the dimension and a basis of the left null space for this particular example.

Solution

(a) The size of matrix A is 3×4 and therefore, the rank of A is at most 3. Furthermore, the matrix A is

$$A=\begin{bmatrix}1&0&0\\2&1&0\\3&4&1\end{bmatrix}\begin{bmatrix}4&2&0&1\\0&0&1&3\\0&0&0&0\end{bmatrix}=\begin{bmatrix}4&2&0&1\\8&4&1&5\\12&6&4&15\end{bmatrix}$$

From the rref of A we observe immediately that the rank is 2 since the last row of the rref matrix is the zero vector. The pivot columns of A can be obtained immediately from R. Let's assume that the first and third column of R are the pivot columns and assume that $x=[x_1 \quad x_2 \quad x_3 \quad x_4]^T$. Therefore, in the system $Ax=0$, the free variables are x_2, x_4. The system $Ax=0$ is reduced to the system $Rx=0$. This consists of the following set of equations:

$$4x_1+2x_2+x_4=0$$
$$x_3+3x_4=0$$

For $x_2=1, x_4=0$ we obtain the special solution $x_{sp1}=[-0.5 \quad 1 \quad 0 \quad 0]^T$. For $x_2=0, x_4=1$ we obtain the special solution $x_{sp2}=[-0.25 \quad 0 \quad -3 \quad 1]^T$. Both of them are perpendicular to the rows of A. This is a universal result, i.e., the solutions to the system $Ax=0$ are always perpendicular to the rows of A as seen directly from the equation $Ax=0$.

(b) As explained in detail in Problem 1, the left null space basis consists of the last $m-r$ rows of the matrix E which is given by the form $EA=R$. In this question we can find E from the operations imposed on A and not by finding the inverse of E^{-1}. At that stage you should be able to do this without help. The sequence of operations on A for the goal of obtaining the rref is equivalent of multiplying A from the left with the matrix

$$E=\begin{bmatrix}1&0&0\\-2&1&0\\5&-4&1\end{bmatrix}$$

The dimension of the left null space for this example is 1, since there is only one zero row in the rref of the original matrix. A basis of the let null space consists of the last $m-r=3-2=1$ rows of E, i.e., it can be the vector $\begin{bmatrix}5\\-4\\1\end{bmatrix}$.

Problem 5

(a) Consider matrices A and B. Explain why every vector in the column space of AB is also in the column space of A. This will tell us an important fact: $\text{rank}(AB)\le \text{rank}\,A$.

(b) Consider matrices A and B. Explain why every vector in the null space of B is also in the null space of AB. Is this also true for every vector in the null space of A or is there an example where it's not true?

Solution

(a) By matrix multiplication, every column vector of AB is a linear combination of column vectors of A. Therefore, every vector in the column space of AB is also in the column space of A.

(b) Suppose a vector x is in the null space of B, then we get $Bx=0$. By matrix multiplication, $ABx = A(Bx) = A\cdot 0 = 0$, therefore x is also in the null space of AB. This is not true for every vector in the null space of AB. Take $A=\begin{bmatrix} a & 0 \\ 2a & 0 \end{bmatrix}$, $B=\begin{bmatrix} b & c \\ d & e \end{bmatrix}$ with $a,c\neq 0$, and $x=\begin{bmatrix} 0 & 1 \end{bmatrix}$. Then $AB=\begin{bmatrix} a & 0 \\ 2a & 0 \end{bmatrix}\begin{bmatrix} b & c \\ d & e \end{bmatrix}=\begin{bmatrix} ab & ac \\ 2ab & 2ac \end{bmatrix}$. We see that $Ax=0$, but x is not in the null space of AB. This is because $ABx=\begin{bmatrix} ab & ac \\ 2ab & 2ac \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} ac \\ 2ac \end{bmatrix}\neq\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Problem 6

Suppose you have applied elimination to A and you have reached $R=\mathrm{rref}(A)$. From looking at R, how would you be able to describe vectors that span the column space of A?

Solution

Since A and its reduced form R have same list of pivot colums, we can easily get the pivot columns of A from R. The column space of A is spanned by those pivot column vectors.

Problem 7

Suppose A is a matrix of size $m\times n$ of rank r. Let $R=\mathrm{rref}(A)$. Assume that the columns of R are rearranged so that its first r columns are the pivot columns.

(a) Find a general form for R.

(b) Find a general form for the **null space matrix** N. This is defined as a matrix whose columns are the special solutions of the system $Ax=0$, and therefore, $AN=\mathrm{O}$ and $RN=\mathrm{O}$, where O is a zero matrix.

Solution

Assume R has r pivot columns and $n-r$ free variables. Each special solution has one free variable equal to 1 and the other free variables are all zero. By setting one free variable equal to 1 and all others equal to 0, we get the values of pivot variables from equation $Rx=0$. In general, suppose that the columns of R are rearranged so that the first r columns of R are the pivot columns. In that case $R=\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times r} & \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}$ where the Os in R indicate zero matrices. The subscripts in the individual matrices reveal their corresponding sizes. Due to the special column rearrangement of R the special solution vectors contain the pivot variables in their first r elements and the free variables in their the last $n-r$ elements. **As already mentioned above**, each special solution has one free variable equal to 1 and the other free variables are all zero. Therefore, the null space matrix N is given by $N=\begin{bmatrix} X_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$ where $X_{r\times(n-r)}$ is an unknown matrix of size $r\times(n-r)$. Knowing that $RN=\mathrm{O}$ we get:

$$RN=\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times r} & \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}\begin{bmatrix} X_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}=\begin{bmatrix} I_{r\times r}\times X_{r\times(n-r)}+F_{r\times(n-r)}\times I_{(n-r)\times(n-r)} \\ \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}=\begin{bmatrix} X_{r\times(n-r)}+F_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}$$

$$X_{r\times(n-r)}+F_{r\times(n-r)}=\mathrm{O}_{r\times(n-r)}\Rightarrow X_{r\times(n-r)}=-F_{r\times(n-r)}$$

Therefore, $N=\begin{bmatrix} -F_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$.

Problem 8

If A is an $m \times n$ rank one matrix $A = uv^T$, where u, v^T are column vectors of length m and n, what vectors are in each of the 4 fundamental subspaces. What are the dimensions of the 4 subspaces?

Solution

$$A = uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

The row space $C(A^T)$ is spanned by v^T and has dimension 1.

The column space $C(A)$ is spanned by u and has dimension 1.

The null space $N(A)$ consists of vectors x of length n such that $v^T x = 0$ with dimension $n-1$.

The left null space $N(A^T)$ consists of vectors x of length m such that $u^T x = 0$ with dimension $m-1$
