# Maths for Signals and Systems

## Problem Sheet 1

**Problem1**
Find two 3-D vectors that are perpendicular to the vector $(1,0,0)$ and to each other.
**Solution**
Two row vectors of size $n \times 1$, $\bar{x} = [x_1, x_2, \ldots, x_n]$ and $\bar{y} = [y_1, y_2, \ldots, y_n]$ are considered perpendicular if their inner product denoted by $\bar{x} \cdot \bar{y}^T = \sum_{i=1}^{n} x_i y_i$ is zero.

In that case we are looking for a vector $(x, y, z)$ such that $1 \cdot x + 0 \cdot y + 0 \cdot z = 0$. Any vector of the form $(0, y, z)$ satisfies that condition. Suppose we choose two vectors $(0, y_1, z_1)$ and $(0, y_2, z_2)$. We also require that these are perpendicular to each other. That means $y_1 z_1 + y_2 z_2 = 0$. The most obvious solution here is $(0, y_1, z_1) = (0,1,0)$ and $(0, y_2, z_2) = (0,0,1)$.

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**Problem2**

(a) Can a matrix of the form below have an inverse? We assume that the elements shown with letters of the alphabet are not zero. Justify your answer.

$$\begin{bmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{bmatrix}$$

(b) What about a matrix of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & 0 & c \\ d & e & 0 \end{bmatrix}$$

**Solution**

(a) In that case the first and third rows are linearly dependent and therefore the rank of the matrix is less than 3. Therefore the matrix doesn't have and inverse.

(b) If we assume that the matrix does **NOT** have an inverse that means that there must be a linear combination of the rows that gives us 0. Therefore for some $(\lambda, \mu, \nu) \neq (0,0,0)$ we have
$\lambda(a,0,b) + \mu(0,0,c) + \nu(d,e,0) = (0,0,0)$.
Therefore $\nu e = 0 \Rightarrow \nu = 0$. That means $\lambda a = 0 \Rightarrow \lambda = 0$ and $\mu c = 0 \Rightarrow \mu = 0$. Therefore the above condition doesn't hold for any non-zero set of coefficients. Therefore the given matrix DOES have an inverse. This is actually the matrix

$$\begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} & b \\ -\frac{d}{ae} & \frac{bd}{ace} & \frac{1}{e} \\ 0 & \frac{1}{c} & 0 \end{bmatrix}.$$

The easiest way to find that is to solve the system of equations which arise from the following matrix multiplication:

$$\begin{bmatrix} a & 0 & b \\ 0 & 0 & c \\ d & e & 0 \end{bmatrix} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = I$$

with unknowns $A, B, C, D, E, F, G, H, I$.

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**Problem 3**

Reduce the system below to the upper triangular form by two row operations:

$$
\begin{aligned}
2x + 3y + z &= 8 \\
4x + 7y + 5z &= 20 \\
-2y + 2z &= 0
\end{aligned}
$$

Circle the pivots. Solve by back substitution for $z, y, x$.

**Solution**

The augmented matrix of the system is

$$
\begin{bmatrix} 2 & 3 & 1 & \vdots & 8 \\ 4 & 7 & 5 & \vdots & 20 \\ 0 & -2 & 2 & \vdots & 0 \end{bmatrix}
$$

In order to eliminate element $(2,1)$ we multiply the first row with 2 and subtract it from the second row. We obtain the matrix:

$$
\begin{bmatrix} 2 & 3 & 1 & \vdots & 8 \\ 0 & 1 & 3 & \vdots & 4 \\ 0 & -2 & 2 & \vdots & 0 \end{bmatrix}
$$

The next step is to eliminate element $(3,1)$ but this element is already zero.

In order to eliminate element $(3,2)$ we multiply the second row with 2 and add it to the third row. We obtain the matrix:

$$
\begin{bmatrix} \textcircled{2} & 3 & 1 & \vdots & 8 \\ 0 & \textcircled{1} & 3 & \vdots & 4 \\ 0 & 0 & \textcircled{8} & \vdots & 8 \end{bmatrix} \text{ with the upper triangular matrix } u = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix}
$$

After these operations we obtain $z = 1,\ y = 1,\ x = 2$. The pivots are shown circled.

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**Problem 4**

Consider the system of equations

$$
\begin{aligned}
2x + 5y + z &= 0 \\
4x + dy + z &= 2 \\
y - z &= 3
\end{aligned}
$$

Which number $d$ forces a row exchange, and what is the triangular system (not singular) for that $d$ ? Which $d$ makes this system singular (no third pivot)?

**Solution**

The augmented matrix of the system is

$$
\begin{bmatrix} 2 & 5 & 1 & \vdots & 0 \\ 4 & d & 1 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & 3 \end{bmatrix}
$$

In order to eliminate element $(2,1)$ we multiply the first row with 2 and subtract it from the second row. We obtain the matrix:

$$
\begin{bmatrix} 2 & 5 & 1 & \vdots & 0 \\ 0 & d-10 & -1 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & 3 \end{bmatrix}
$$

In case $d-10=0$ which gives $d=10$, rows 2 and 3 have to be swapped. In case $d-10=1$ which gives $d=11$ rows 2 and 3 are identical and therefore the matrix is not invertible, i.e., it is singular.

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**Problem 5**
If $(a,b)$ is a multiple of $(c,d)$ with $abcd \neq 0$, show that $(a,c)$ is a multiple of $(b,d)$. This observation yields the conclusion that the matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent columns when it has dependent rows.
**Solution**
Suppose that $(a,b)=\lambda(c,d)$.
In that case $(a,c)=(\lambda c,c)=c(\lambda,1)$. Furthermore, $(b,d)=(\lambda d,d)=d(\lambda,1)$.
Therefore $(a,c)=c(\lambda,1)=\frac{c}{d}d(\lambda,1)=\frac{c}{d}(b,d)$, i.e., $(a,c)$ is a multiple of $(b,d)$.

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**Problem 6**
Consider the matrix

$$A=\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 5 \end{bmatrix}.$$

Find the reduced row echelon form (rref) $R$. Write $A$ in the form $A=ER$. Find the matrix $E$.
**Solution**
In order to eliminate element $(2,1)$ of $A$ we subtract the first row from the second row. We obtain the matrix $A_1$:

$$A_1=\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 5 \end{bmatrix}$$

The above operation is equivalent with multiplying the original matrix $A$ with matrix

$$E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with inverse

$$E_{21}^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In order to eliminate element $(3,1)$ from matrix $A_1$ we multiply the first row of $A_1$ with 2 and subtract it from the third row. We obtain the matrix $A_2$:

$$A_2=\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The above operation is equivalent of multiplying the matrix $A_1$ with matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

with inverse

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

In order to eliminate element $(3,2)$ from matrix $A_2$ we subtract the second row of matrix $A_2$ from the third row. We obtain the matrix $A_3$:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above operation is equivalent of multiplying the matrix $A_2$ with matrix

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

with inverse

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

We continue the elimination process in order to remove elements located within the top right part of the matrix. In order to remove element $(1,2)$ from matrix $A_3$ we subtract the second row of matrix $A_3$ from the first row. This is the last elimination we can do on this matrix. Therefore, we obtain

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above operation is equivalent of multiplying the matrix $A_3$ with matrix

$$E_{12} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with inverse

$$E_{12}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $E_{12}E_{32}E_{31}E_{21}A = R \Rightarrow A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}E_{12}^{-1}R$

$$E = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}E_{12}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

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