1. a) Specify whether the matrix below has an inverse without trying to compute the inverse.

$$R = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

[2]

b) Let $A = \begin{bmatrix} 1 & a & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Determine those values of a for which A is invertible. **[2]**

c) Find the volume of the parallelepiped S formed by the triple of vectors in R^3, $x = (1,1,1)^T$, $y = (2,3,4)^T$, $z = (1,1,5)^T$. **[2]**

d) An $n \times n$ matrix A is called skew-symmetric if $A^T = -A$. Show that if A is skew-symmetric and n is an odd positive integer, then A is not invertible. **[4]**

e) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and assume that $\det(A) = 10$. Find $\det(5A)$, $\det(3A^{-1})$, $\det(3A^3)$, $\det[2(A^T)^{-1}]$, and $\det(B)$ with $A = \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$. **[4]**

f) Find the solution of the following system of equations using QR decomposition.

$$x - 2y - 2z = 3$$
$$-x + 2y + 3z = 1$$
$$2x - 2y - 2z = -2$$

[6]

2. a) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 3 & 3 \\ 2 & 1 & -2 & 0 & 3 & 4 \\ 1 & 0 & -1 & -2 & 0 & 1 \\ 3 & 2 & -3 & 2 & 6 & 7 \end{bmatrix}$$

(i) By using elimination find the dimension and a basis of the row space of A, $R(A)$. **[2]**

(ii) Find the dimension and a basis of the nullspace of A, $N(A)$. **[2]**

(iii) Find the dimension and a basis of the column space of A, $C(A)$. **[2]**

(iv) Find the dimension and a basis of the left nullspace. **[2]**

b) Mark each statement (i)-(v) True or False. Justify your answer. Let S be a set of m vectors in R^n.

(i) If $m > n$ then the vectors in S are linearly independent. **[1]**

(ii) If the zero vector is in S, then the vectors in S are linearly dependent. **[1]**

(iii) If the vectors in S are linearly independent and T is a subset of S, then the vectors in T are linearly independent. **[1]**

(iv) If the vectors in T are linearly dependent and T is a subset of S, then the vectors in S are linearly dependent. **[1]**

(v) The linear system $Ax = b$ has a unique solution if and only if the column vectors of A are linearly independent. **[2]**

c) We are seeking to fit the 5 two-dimensional points $(-2,0), (-1,0), (0,1), (1,1)$ onto a straight line.

(i) Give the system of equations that we must solve in order to achieve the above requirement. Explain why the system doesn't have a solution. **[2]**

(ii) Find an approximate solution of the system using the least squares approach and give the equation for the required straight line. **[2]**

(iii) Calculate the magnitude of the error of the approximation. **[2]**

3. a) Consider a matrix A with characteristic polynomial $\lambda^5 - 10\lambda^4 + 3\lambda^3 + \lambda^2 - 2\lambda + 7$. Is A invertible? Justify your answer. **[2]**

b) Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of a matrix A corresponding to an eigenvalue of 3 and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to an eigenvalue of -2. Compute $A^3 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

[2]

c) Consider the matrix A:

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Determine if A is diagonalizable, and if so, diagonalize it. **[6]**

d) Consider the matrices A and B shown below.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Both matrices have characteristic polynomial $p_A(\lambda) = p_B(\lambda) = -(\lambda - 1)(\lambda + 2)^2$.

(i) Find all eigenvectors of matrix A. **[3]**

(ii) Find all eigenvectors of matrix B. **[3]**

(iii) State which of the above matrices A, B are diagonalizable. **[2]**

(iv) Diagonalize the matrix or matrices, if any, stated in (iii) above. **[2]**