1. a) The first two columns are dependent and therefore the matrix is not invertible. **[2]**

b) $\det(A) = 1 - a$ and therefore, in order for A to be invertible then $a \neq 1$. **[2]**

c) The required volume is $\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 1 & 5 \end{bmatrix}\right) = 4$ **[2]**

d) $\det(A^T) = \det(A) = \det(-A) = (-1)^n \det(A)$. If n is an odd positive integer then $(-1)^n = -1$ and therefore, $\det(A) = -\det(A) \Rightarrow \det(A) = 0$. This means that the matrix is not invertible.
[4]

e) $\det(5A) = 5^3 \det(A) = 1250$

$\det(3A^{-1}) = \dfrac{3^3}{\det(A)} = \dfrac{27}{10}$

$\det(3A^3) = 3^3 [\det(A)]^3 = 27 \times 1000 = 2700$

$\det[2(A^T)^{-1}] = 2^3 \det[(A^T)^{-1}] = 2^3 \dfrac{1}{\det(A^T)} = \dfrac{2^3}{\det(A)} = \dfrac{8}{10}$

$\det(B) = -\det(A) = -10$ since B is obtained by transposing A and making a single row swap. **[4]**

f) We select the first orthogonal direction to be $A = a = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\|A\| = \sqrt{6}$. Therefore,

$q_1 = \dfrac{A}{\|A\|} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$. The second direction is:

$$B = b - \frac{AA^T}{A^T A} b = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} =$$

$$\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -8 \\ 8 \\ -16 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}$$

$$\|B\| = \frac{1}{6}\sqrt{3 \times 16} = \frac{4\sqrt{3}}{6}$$

$$q_2 = \frac{B}{\|B\|} = \frac{6}{4\sqrt{3}} \frac{1}{6}\begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{BB^T}{B^T B} = \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\frac{BB^T}{B^T B}c = \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$$

$$\frac{AA^T}{A^T A} = \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

$$\frac{AA^T}{A^T A}c = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -9 \\ 9 \\ -18 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}$$

$$\frac{BB^T}{B^T B}c + \frac{AA^T}{A^T A}c = \frac{1}{2}\begin{bmatrix} -5 \\ 5 \\ -4 \end{bmatrix}$$

$$c - \frac{BB^T}{B^T B}c - \frac{AA^T}{A^T A}c = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -5 \\ 5 \\ -4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\|c\| = \frac{1}{2}\sqrt{2}$$

$$q_3 = \frac{c}{\|c\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, $Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$q_1^T a = \sqrt{6}$, $q_1^T b = -\frac{8}{\sqrt{6}}$, $q_1^T c = -\frac{9}{\sqrt{6}}$, $q_2^T b = \frac{2}{\sqrt{3}}$, $q_2^T c = \frac{3}{\sqrt{3}}$, $q_3^T c = \frac{1}{\sqrt{2}}$

$$R = \begin{bmatrix} \sqrt{6} & \frac{-8}{\sqrt{6}} & \frac{-9}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{3}} & \frac{3}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The solution of the system $Fx = b$ is given now through the system $Rx = Q^T b$

[6]

2. a)

(i) The sequential steps of elimination give the following intermediate matrices:

$$A=\begin{bmatrix}1 & 1 & -1 & 2 & 3 & 3\\ 2 & 1 & -2 & 0 & 3 & 4\\ 1 & 0 & -1 & -2 & 0 & 1\\ 3 & 2 & -3 & 2 & 6 & 7\end{bmatrix} \text{ and then } \begin{bmatrix}1 & 1 & -1 & 2 & 3 & 3\\ 2 & 1 & -2 & 0 & 3 & 4\\ 1 & 0 & -1 & -2 & 0 & 1\\ 0 & -1 & 0 & -4 & -3 & -2\end{bmatrix},$$

$$\begin{bmatrix}1 & 1 & -1 & 2 & 3 & 3\\ 2 & 1 & -2 & 0 & 3 & 4\\ 0 & -1 & 0 & -4 & -3 & -2\\ 0 & -1 & 0 & -4 & -3 & -2\end{bmatrix}, \begin{bmatrix}1 & 1 & -1 & 2 & 3 & 3\\ 2 & 1 & -2 & 0 & 3 & 4\\ 0 & -1 & 0 & -4 & -3 & -2\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}1 & 1 & -1 & 2 & 3 & 3\\ 0 & -1 & 0 & -4 & -3 & -2\\ 0 & -1 & 0 & -4 & -3 & -2\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \text{ and finally } R=\begin{bmatrix}1 & 1 & -1 & 2 & 3 & 3\\ 0 & -1 & 0 & -4 & -3 & -2\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

We know that $EA=R$ with

$$E=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & =1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -3 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 1 & -1 & 1 & 0\\ -2 & 0 & -1 & 1\end{bmatrix}.$$

The row space has dimension 2. A basis of the row space can be formed by the first two rows of R. **[2]**

(ii) The nullspace has dimension 6-2=4. We choose the free variables to be the ones which correspond to columns 3 to 6. We then find the four special solutions. These consist a basis of the nullspace. They are: $[1 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T$, $[2 \;\; -4 \;\; 0 \;\; 1 \;\; 0 \;\; 0]^T$, $[0 \;\; -3 \;\; 0 \;\; 0 \;\; 1 \;\; 0]^T$, $[-1 \;\; -2 \;\; 0 \;\; 0 \;\; 0 \;\; 1]^T$ **[2]**

(iii) The column space has the same dimension as the row space, i.e., 2. We can choose any two independent columns for example the first two, to form a basis for the column space. **[2]**

(iv) $EA=R$ and the last two rows of R are zero and therefore, the last 2 rows of E form a basis of the left nullspace which has dimension is 2. **[2]**

b)

(i) FALSE since the maximum number of independent vectors in R^n is n. **[1]**

(ii) TRUE the zero vector is dependent to all vectors. **[1]**

(iii) TRUE since if two vectors in T were dependent then two vectors in S would be dependent. **[1]**

(iv) TRUE obvious **[1]**

(v) FALSE if the rows are more than the columns and columns are independent the system might not have a solution. **[2]**

c)

(i) We have a set of equations

$$C - 2D = 0$$
$$C - D = 0$$
$$C = 1$$
$$C + D = 1$$

and therefore the system is

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The system doesn't have a solution since the solutions that is obtained from 2 of the equations doesn't satisfy the rest. **[2]**

(ii) Instead of solving the system $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ we solve the system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow C = \frac{7}{10}, D = \frac{4}{5}$$

The straight line is $7/10 + 4t/5$. **[2]**

(iii) The error vector is

$$e = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{10} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{9}{10} \\ \frac{1}{10} \\ \frac{3}{10} \\ \frac{1}{2} \end{bmatrix}$$

[2]

3. a) For $\lambda = 0$ the characteristic polynomial is 7. Therefore, 0 is not a root of the characteristic polynomial in which case the matrix is invertible. **[2]**

b) In that case $S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $S^{-1} = -\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

Therefore, $A = S\Lambda S^{-1} = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$, $A^3 = S\Lambda^3 S^{-1} = \begin{bmatrix} -43 & 70 \\ -35 & 62 \end{bmatrix}$,

$A^3\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -43 & 70 \\ -35 & 62 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 38 \\ 46 \end{bmatrix}$ **[2]**

c) Consider the matrix A: The eigenvalues are 4,4,2,2 and therefore the matrix has repeated eigenvalues. It might or might not be diagonalizable. The eigenvectors are $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ which are independent. Therefore, the matrix is diagonalizable. **[6]**

d)

(i) For matrix A we have

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{matrix} x+3y+3z = x \\ -3x-5y-3z = y \\ 3x+3y+z = z \end{matrix} \Rightarrow \begin{matrix} 3y+3z = 0 \\ -3x-6y-3z = 0 \\ 3x+3y = 0 \end{matrix} \Rightarrow$$

$$\begin{matrix} y+z=0 \\ x+y=0 \end{matrix} \Rightarrow \begin{matrix} z=-y \\ x=-y=z \end{matrix}$$

An eigenvector can be $[1 \quad -1 \quad 1]^T$

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{matrix} x+3y+3z = -2x \\ -3x-5y-3z = -2y \\ 3x+3y+z = -2z \end{matrix} \Rightarrow \begin{matrix} 3x+3y+3z = 0 \\ -3x-3y-3z = 0 \\ 3x+3y+3z = 0 \end{matrix} \Rightarrow$$

$$x+y+z=0 \Rightarrow z=-x-y$$

The eigenvectors are $[x \quad y \quad -x-y]^T = x[1 \quad 0 \quad -1]^T + y[0 \quad 1 \quad -1]^T$ **[3]**

(ii) For matrix B we have

$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{matrix} 2x+4y+3z = x \\ -4x-6y-3z = y \\ 3x+3y+z = z \end{matrix} \Rightarrow \begin{matrix} x+4y+3z = 0 \\ -4x-7y-3z = 0 \\ 3x+3y = 0 \end{matrix} \Rightarrow$$

$$\begin{matrix} y+z=0 \\ x+y=0 \end{matrix} \Rightarrow \begin{matrix} z=-y \\ x=-y=z \end{matrix}$$

An eigenvector can be $[1 \quad -1 \quad 1]^T$

$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{matrix} 2x+4y+3z = -2x \\ -4x-6y-3z = -2y \\ 3x+3y+z = -2z \end{matrix} \Rightarrow \begin{matrix} 4x+4y+3z = 0 \\ -4x-4y-3z = 0 \\ 3x+3y+3z = 0 \end{matrix} \Rightarrow$$

$$\begin{matrix} x+y+z=0 \\ 4x+4y+3z=0 \end{matrix} \Rightarrow \begin{matrix} x+y=0 \\ z=0 \end{matrix}$$

An eigenvector can be $[1 \quad -1 \quad 0]^T$ **[3]**

(iii) A is diagonalizable because it has 3 independent eigenvectors but B is not. **[2]**

(iv) $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ $S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ and $A = S\Lambda S^{-1}$ **[2]**