1. a) Suppose A is a matrix of size $m \times n$ of rank r. Let $R = \text{rref}(A)$ be the row reduced echelon form of A.

(i) If we assume that the first r columns of R are the pivot columns, then R can be written as $R = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$. In that representation B, C, D, E are sub-matrices. Give the form of all matrices B, C, D, E and their dimensions. Justify your answer. **[2]**

(ii) Find a general form for the null space matrix N. This is defined as a matrix whose columns are the special solutions of the system $Ax = 0$. In that representation 0 is a column vector with zero elements. **[2]**

(iii) Suppose you know that the 3×4 matrix A has the vector $\begin{bmatrix} 4 & 2 & 0 & 2 \end{bmatrix}^T$ as a basis for its null space. What is the row reduced echelon form R of A? **[2]**

(iv) Show that a basis for the left null space of a rectangular matrix A of size $m \times n$ consists of the last $m - r$ rows of the corresponding elimination matrix E. **[2]**

b) Suppose that the matrix A is the product

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Give a basis for the null space of A and a basis for the row space of A. **[2]**

(ii) Express row 3 of A as a combination of your basis vectors in your answer to (i). **[2]**

(iii) What is the dimension of the null space of A^T? Justify your answer. **[2]**

c) Assume that the matrix A is invertible and permutations are not required in elimination.

(i) Show that the product of pivots of A^{-1} is related to the product of pivots of A. **[2]**

(ii) Apply block elimination to the $2n \times 2n$ block-matrix

$$M = \begin{bmatrix} A & I \\ -I & \text{O} \end{bmatrix}$$

The elements A, I, O are sub-matrices of size $n \times n$. The sub-matrix I is the identity matrix and the sub-matrix O is a matrix with zero elements. **[2]**

(iii) Find the determinant of M by exchanging the first n columns of M with the last n columns and using properties of determinants. **[2]**

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2. a) Consider the system of equations $Ax = b$, where A is a matrix of size $m \times n, m > n$ and the vector b does not belong to the column space of the matrix A. In an attempt to find an approximate solution to the above system we seek for $\hat{x}$ that satisfies the system $A\hat{x} = p$ where p is the projection of b onto the column space of A.

(i) Give the matrix P that projects b onto the column space of A for the general case where A is a matrix of size $m \times n, m > n$ and $A^T A$ is invertible. Give the matrix P for the specific case where A is a square invertible matrix. **[2]**

(ii) Show that P is symmetric and furthermore, that $P^2 = P$. **[2]**

(iii) Show that the projection of b onto to the column space of A is zero if b is perpendicular to the column space of A. **[1]**

(iv) Give the error of the approximation and show that the error belongs to the null space of A^T. **[1]**

b) Consider the following matrix:

$$P = \frac{1}{14}\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & 9 \end{bmatrix}$$

(i) Is P a projection matrix? Justify your answer. **[2]**

(ii) What subspace does P project onto? **[1]**

(iii) What is the distance from that subspace to $b = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T$? **[1]**

(iv) What are the eigenvalues of P? **[2]**

(v) Is P diagonalizable? By observing the structure of P give the type of decomposition that P is amenable to. **[2]**

c) We are seeking to fit the 5 two-dimensional points $(-2,0), (-1,0), (0,1), (1,1), (2,1)$ onto a straight line.

(i) Give the system of equations that we must solve in order to achieve the above requirement. Explain why the system doesn't have a solution. **[2]**

(ii) Find an approximate solution of the system using the least squares approach and give the equation for the required straight line. **[2]**

(iii) Calculate the magnitude of the error of the approximation. **[2]**

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3. a) Consider the matrix A:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(i) Find the null space of A. **[2]**

(ii) Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Calculate its eigenvalues. **[2]**

(iii) Find the relationship between the matrix $A^T A$ and B and calculate the eigenvalues of the matrix $A^T A$. **[2]**

(iv) For the Singular Value Decomposition of matrix A, $A = U\Sigma V^T$, find the nonzero entries in the diagonal matrix Σ and one column of the matrix V. **[2]**

b) Consider a square matrix Z that has orthonormal columns.

(i) Show that Z satisfies $Z^T Z = I$. **[2]**

(ii) Show that $\|Zx\|^2 = \|x\|^2$ where x is any vector. **[2]**

c) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

[8]

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