Maths for Signals and Systems Exam 2014-Solutions

1. a) (i) I is the identity matrix, O are zero matrices and F is a matrix that is related to the special solutions of the system.
The dimensions of the individual matrices are given in the subscripts $R=\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times r} & \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}$ The subscripts in the individual matrices reveal their corresponding sizes.

(ii) Due to the special column rearrangement of R the special solution vectors contain the pivot variables in their first r elements and the free variables in their the last $n-r$ elements. **As already mentioned above**, each special solution has one free variable equal to 1 and the other free variables are all zero. Therefore, the null space matrix N is given by $N=\begin{bmatrix} X_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$ where $X_{r\times(n-r)}$ is an unknown matrix of size $r\times(n-r)$.
Knowing that $RN=\mathrm{O}$ we get:

$$RN=\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times r} & \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}\begin{bmatrix} X_{r\times(n-r)} \\ I_{r\times(n-r)} \end{bmatrix}$$

$$=\begin{bmatrix} I_{r\times r}\times X_{r\times(n-r)}+F_{r\times(n-r)}\times I_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}=\begin{bmatrix} X_{r\times(n-r)}+F_{r\times(n-r)} \\ \mathrm{O}_{(m-r)\times(n-r)} \end{bmatrix}$$

$$X_{r\times(n-r)}+F_{r\times(n-r)}=\mathrm{O}_{r\times(n-r)} \Rightarrow X_{r\times(n-r)}=-F_{r\times(n-r)}$$

Therefore, $N=\begin{bmatrix} -F_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$.

(iii) We assume that the echelon form is obtained without any permutations. In case of a 3×4 matrix, the maximum rank is 3. In that case we are given that the dimension of the null space is 1. Since the rows of the matrix are 4-dimensional, we know immediately that the dimension of the row space is 3. Therefore, the rank of the matrix is 3. In that case the echelon matrix must be of the form $R=\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$. The null space is obtaine by looking for random vectors x, for which $Ax=0\Rightarrow ERx=0$. This implies $Ax=0\Rightarrow ERx=0$ the $Rx=0$, since the matrix E is a square, full rank matrix.

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 4+2a=0\Rightarrow a=-2 \\ 2+2b=0\Rightarrow b=-1 \\ 2c=0\Rightarrow c=0 \end{matrix} \Rightarrow R=\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(iv) We know that $EA=R$ where $E=\prod_{ij} E_{ij}$ is the product of all elimination matrices used in the procedure. If the rank of matrix A is r then the last $m-r$ rows of R are zero rows. Therefore, from the equation $EA=R$ we see that each of the last $m-r$ rows of E multiplied with A from the left gives a zero row vector. This verifies the fact that the last $m-r$ rows of E belong to the left null space, since they satisfy the relationship $x^T A=0^T$. Due to the method that we use to construct E, it can be shown easily that E is a full rank matrix (rank is m) and therefore its last $m-r$ rows are independent. Since these rows belong to the left null space and knowing that the left null space has dimension $m-r$, we can say that the last $m-r$ rows of E form a basis of the left null space.

b) (i) The echelon form has two pivots. Therefore, the rank of the matrix is 2. The rows are 4-dimensional and therefore, we have 2 free variables. We solve the system $\begin{bmatrix}1 & 0 & 3 & 4\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=0$ for $z=1, w=0$ and $z=0, w=1$.

For $z=1, w=0$ we have $x+3=0 \Rightarrow x=-3$ and $y+z=0 \Rightarrow y=-1$.
For $z=0, w=1$ we have $x+4=0 \Rightarrow x=-4$ and $y=0$.

Therefore, the special solutions are $\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-4\\0\\0\\1\end{bmatrix}$ and they form a basis for the nullspace.

The row space does not change with elimination and therefore, any two independent rows of the echelon matrix, for example rows 1 and 2, form a basis for the row space.

(ii) 5(row1)+4(row2)

(iii) A has rank 2 and A^T is 4 by 3 so its null space has dimension 3-2=1.

c) (i) The pivots of A^{-1} are equal to 1/(pivots of A) because $\det A^{-1} = 1/(\det A)$.

(ii) Multiply row 1 by A^{-1} and add to row 2 to obtain $\begin{bmatrix}A & I\\ \mathrm{O} & A^{-1}\end{bmatrix}$

(iii) The determinant is +1. Exchange the first n columns with the last n. This produces a factor $(-1)^n$ and leaves $\begin{bmatrix}I & A\\ \mathrm{O} & -I\end{bmatrix}$ which is triangular with determinant $(-1)^n$. Then $(-1)^n(-1)^n = +1$.

2. a) (i) $P = A(A^TA)^{-1}A^T$

(ii) A^TA is symmetric and therefore $(A^TA)^{-1}$ is symmetric. (To prove this we use the property $(A^{-1})^T = (A^T)^{-1}$.)

$P^T = [A(A^TA)^{-1}A^T]^T = (A^T)^T[(A^TA)^{-1}]^T A^T = A(A^TA)^{-1}A^T = P$

$P^2 = [A(A^TA)^{-1}A^T][A(A^TA)^{-1}A^T] = [A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T] = P$

If A is square and invertible its column space is the entire n-dimensional space and therefore the projection of b onto A should be b. In that case $P = AA^{-1}(A^T)^{-1}A^T = I$.

(iii) If b is perpendicular to the column space of A then $Pb = AA^{-1}(A^T)^{-1}A^Tb = 0$.

(iv) $e = b - Pb$, $A^Te = A^Tb - A^Tp = A^Tb - A^TPb = 0$

b) (i) **The projection matrix P is of the form $P = A(A^TA)^{-1}A^T$ with A being the column vector $[1 \quad 2 \quad -3]^T$. Therefore, it projects onto the column space of A which is the line $c[1 \quad 2 \quad -3]^T$.**

(ii) **The error is** $e = b - Pb = \frac{1}{7}\begin{bmatrix} -6 \\ 9 \\ 4 \end{bmatrix}$ **and the distance is** $\|e\| = \frac{\sqrt{133}}{7}$.

(iii) Since P projects onto a line, its three eigenvalues are 0,0,1. Since P is symmetric, it has a full set of (orthogonal) eigenvectors, and is then diagonalizable.

c) (i) We have a set of equations

$$C - 2D = 0$$
$$C - D = 0$$
$$C = 1$$
$$C + D = 1$$
$$C + 2D = 1$$

and therefore the system is

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The system doesn't have a solution since the solutions that is obtained from 2 of the equations doesn't satisfy the rest.

(ii) The projection matrix is

$$\begin{bmatrix} 3/5 & 2/5 & 1/5 & 0 & -1/5 \\ 2/5 & 3/10 & 1/5 & 1/10 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/10 & 1/5 & 3/10 & 2/5 \\ -1/5 & 0 & 1/5 & 2/5 & 3/5 \end{bmatrix}$$

and the projection vector is

$$\begin{bmatrix} 0 \\ 3/10 \\ 3/5 \\ 9/10 \\ 6/5 \end{bmatrix}$$

Approximate solution is $C = 6/10$ and $D = 3/10$. Straight line is $6/10 + 3t/10$.

(iii) error vector is

$$e = b - p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3/10 \\ 3/5 \\ 9/10 \\ 6/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/10 \\ 2/5 \\ 1/10 \\ -1/5 \end{bmatrix}$$

3. a) (i) By solving the system $Ax = 0$, it is straightforward to see that the null space has dimension 1 and its basis is the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(ii) Matrix B is singular. All rows are identical and therefore the row space is of dimension 1. Therefore, 3 out of 4 eigenvalues of B must be 0. The remaining non-zero eigenvalue can be found from the trace of B and it is equal to 4. Therefore, the eigenvalues of B are 4,0,0,0.

(iii) $A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} = 4I - B$

If x is an eigenvector of B with eigenvalue λ, then

$Bx = \lambda x \Rightarrow 4x - Bx = 4x - \lambda x \Rightarrow (4I - B)x = (4 - \lambda)x \Rightarrow A^T A x = (4 - \lambda)x$

Therefore, the eigenvalues of $A^T A$ are obtained from the eigenvalues of B, by reversing the sign and adding 4. Thus, the eigenvalues of $A^T A$ are 0,4,4,4.

(iv) The non-zero singular values of matrix A, are the square roots of the eigenvalues of $A^T A$. Therefore, these are 2,2,2. The matrix $A^T A$ is diagonalized through the formula $A^T A = V \Sigma V^T$ where $A = U \Sigma V^T$. The matrix V has the eigenvectors of $A^T A$ in its columns.

The eigenvector of $A^T A$ that corresponds to 0 is of the form $\begin{bmatrix} x & x & x & x \end{bmatrix}^T$ with magnitude $\sqrt{4x^2}$. If we look for an orthonormal eigenvector then $\sqrt{4x^2} = 1 \Rightarrow x = \frac{1}{2}$.

Therefore, a column of V is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T$.

b) We select the first orthogonal direction to be $A = a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\|A\| = \sqrt{2}$. Therefore,

$q_1 = \frac{A}{\|A\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$. The second direction is:

$$B = b - \frac{AA^T}{A^T A} b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\|B\| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$$

$$q_2 = \sqrt{\frac{2}{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\frac{BB^T}{B^TB} = \frac{2}{3}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1/4 & -1/4 & 1/2 \\ -1/4 & 1/4 & -1/2 \\ 1/2 & -1/2 & 1 \end{bmatrix}$$

$$\frac{BB^T}{B^TB}c = \frac{2}{3}\begin{bmatrix} 1/4 & -1/4 & 1/2 \\ -1/4 & 1/4 & -1/2 \\ 1/2 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1/4 \\ -1/4 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix}$$

$$\frac{AA^T}{A^TA}c = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\frac{BB^T}{B^TB}c + \frac{AA^T}{A^TA}c = \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$c - \frac{BB^T}{B^TB}c - \frac{AA^T}{A^TA}c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\|c\| = \frac{2}{\sqrt{3}}$$

$$q_3 = \frac{c}{\|c\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$q_1^T a = \sqrt{2}$, $q_1^T b = \frac{1}{\sqrt{2}}$, $q_1^T c = \frac{1}{\sqrt{2}}$, $q_2^T b = \frac{3}{\sqrt{6}}$, $q_2^T c = \frac{1}{\sqrt{6}}$, $q_3^T c = \frac{2}{\sqrt{3}}$

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$