Imperial College London

# Maths for Signals and Systems

## Linear Algebra in Engineering

### Lectures 16-17, Tuesday 18th November 2014

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## Symmetric matrices

- In this lecture we will be interested in symmetric matrices.
- Consider a matrix $A = A^T$.
- The eigenvalues are real.
- The eigenvectors can be chosen to be perpendicular. If we also choose them to have a magnitude of 1, then the eigenvectors can be chosen to form an orthonormal set of vectors.
- For a random matrix with independent eigenvectors we have $A = S\Lambda S^{-1}$.
- For a symmetric matrix with orthonormal eigenvectors we have

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T \text{ (symmetric)}$$

## Symmetric matrices

- Consider $Ax = \lambda x$.
- If we take complex conjugate in both sides we get
$$(Ax)^* = (\lambda x)^* \Rightarrow A^*\ x^* = \lambda^*\ x^*$$
- If $A$ is real then $Ax^* = \lambda^*\ x^*$. Therefore, if $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $x$ then $\lambda^*$ is an eigenvalue of $A$ with corresponding eigenvector $x^*$.

## Symmetric matrices

- **Problem:** Why are the eigenvalues of a symmetric matrix real?

We proved that if $A$ is real then $Ax^* = \lambda^* x^*$.

If we take transpose in both sides we get

$$x^{*T}A^T = \lambda^* x^{*T} \Rightarrow x^{*T}A = \lambda^* x^{*T}$$

We now multiply both sides from the right with $x$ and we get

$$x^{*T}Ax = \lambda^* x^{*T}x$$

We take now $Ax = \lambda x$. We now multiply both sides from the left with $x^{*T}$ and we get

$$x^{*T}Ax = \lambda x^{*T}x.$$

From the above we see that $\lambda x^{*T}x = \lambda^* x^{*T}x$ and since $x^{*T}x \neq 0$, we see that $\lambda = \lambda^*$.

# Complex matrices: Which complex matrices are “good”?

**(meaning real eigenvalues and orthogonal eigenvectors)**

- Consider $Ax = \lambda x$ with $A$ possibly complex.
- If we take complex conjugate in both sides we get
$$(Ax)^* = (\lambda x)^* \Rightarrow A^*\, x^* = \lambda^*\, x^*$$
- If we take transpose in both sides we get
$$x^{*T} A^{*T} = \lambda^* x^{*T}$$

We now multiply both sides from the right with $x$ we get
$$x^{*T} A^{*T} x = \lambda^* x^{*T} x$$

We take now $Ax = \lambda x$. We now multiply both sides from the left with $x^{*T}$ and we get
$$x^{*T} A x = \lambda x^{*T} x.$$

From the above we see that if $A^{*T} = A$ then $\lambda x^{*T} x = \lambda^* x^{*T} x$ and since $x^{*T} x \neq 0$, we see that $\lambda = \lambda^*$.

# Eigenvalue sign

- We proved that the eigenvalues of a symmetric matrix are real.
- It can be proven that the signs of the pivots are the same as the signs of the eigenvalues.
- Just to remind you:

  Product of pivots=Product of eigenvalues=Determinant

# Positive definite matrices

- These are matrices with all their eigenvalues real and positive!
- Therefore, the pivots are positive and the determinant is positive.
- Positive determinant doesn't guarantee positive definiteness. All sub-determinants have to be positive.
- **Example:** Consider the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

Pivots are 5 and 11/5. This comes directly from the fact that the product of pivots equals the determinant.

Eigenvalues are obtained from:

$$(5-\lambda)(3-\lambda)-4=0 \Rightarrow \lambda^2-8\lambda+11=0$$

$$\lambda_{1,2}=\frac{8\pm\sqrt{64-44}}{2}=\frac{8\pm\sqrt{20}}{2}=4\pm\sqrt{5}$$

# Complex vectors and matrices

- Consider a complex column vector $z = [z_1 \quad z_2 \quad \cdots \quad z_n]^T$.
- It's length is ${z^*}^T z = \sum_{i=1}^{n} |z_i|^2$.
- When we both transpose and conjugate we can use the symbol $z^H = {z^*}^T$ (Hermitian).
- Inner product of 2 complex vectors is ${y^*}^T x = y^H x$.
- For complex matrices the symmetry is defined as ${A^*}^T = A$. These are called Hermitian matrices.
- They have real eigenvalues and perpendicular unit eigenvectors. If these are complex we check their length using ${q_i^*}^T q_i$ and also ${Q^*}^T Q = I$.
- **Example:** Consider the matrix

$$A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

# The Fourier matrix

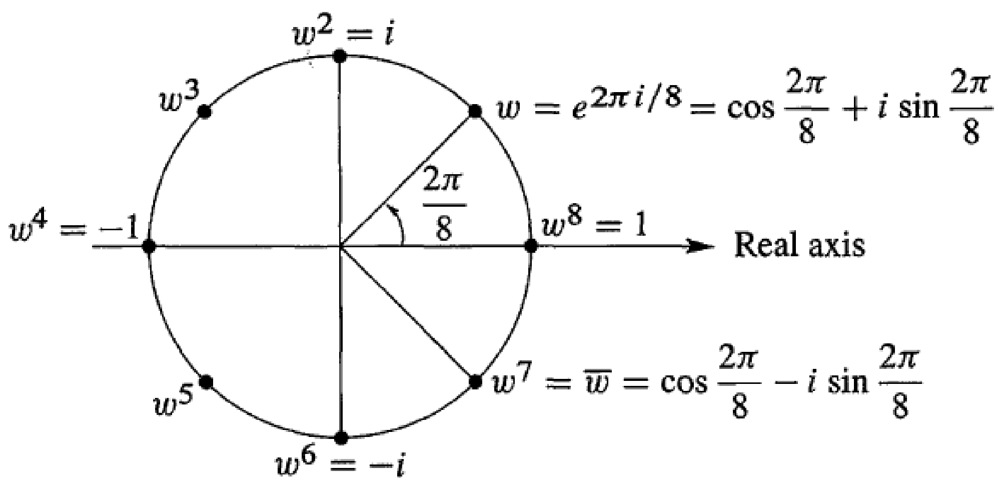
- The $n \times n$ Fourier matrix is defined as:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & \dots & w^{(n-1)} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(n-1)} & w^{2(n-1)} & \dots & w^{(n-1)(n-1)} \end{bmatrix}$$

- In this matrix we will number the first row and column with 0.
- We define $w = e^{i\frac{2\pi}{n}}$. For $w$ is preferable to use polar representation.
- $F_n(i,j) = w^{ij}$.
- We must stress out that it is better to use the notation $w_n$ instead of $w$.
- I have avoided this notation to make things look simpler.

# The Fourier matrix

- The parameter $w = e^{i\frac{2\pi}{n}}$ lies on the unit circle shown below. The case depicted below refers to $n = 8$ where the points $w^m, m = 0, \dots, 7$ of the second row (row 1) of the Fourier matrix are shown.




- We must stress out that the Fourier matrix is totally constructed out of numbers of the form ${w_n}^k$.

# The Fourier matrix for $n = 4$

- The parameter $w_4 = e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$.
- The quantities inside Fourier matrix are $1, i, i^2, i^3, i^4, i^6, i^9$.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

- The columns of this matrix are orthogonal.
- Remember that the inner product of 2 complex vectors is $y^{*T}x = y^H x$!

# The Fourier matrix for $n = 4$

- I can show that the columns are orthogonal but they are not orthonormal.
- I can fix this by dividing the Fourier matrix with the length of the rows (columns). In this case it is 2. Therefore, I can write:

$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

- We know that ${F_4}^H F_4 = I$.

# The Fast Fourier Transform

- We can prove that there is a connection between $F_{2n}$ and $F_n$.
- This is expected from the fact that ${w_{2n}}^2 = w_n$. It can be shown that:

diagonal matrix $D_n$

permutation matrix $P_{2n}$

$$[F_{2n}] = \begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix} \begin{bmatrix} F_n & \mathbf{0} \\ \mathbf{0} & F_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$[F_{2n}]$: yields to $(2n)^2$ multiplications

$\begin{bmatrix} F_n & \mathbf{0} \\ \mathbf{0} & F_n \end{bmatrix}$+influence of $D_n$: yield to $2 \times (n)^2 + n$ multiplications!!! ($I_n$ and $P_{2n}$ don't contribute to multiplications)

# The Fast Fourier Transform

- In the previous analysis the matrix $D_n$ is defined as:

$$D_n = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{n-1} \end{bmatrix}$$

- We start with $(2n)^2$ multiplications and manage to reduce them to $2 \times (n)^2 + n$ multiplications!!

## The Fast Fourier Transform

- The next step is to break the $F_n$ down. As you see, I use the above idea recursively!

$$[F_{2n}] = \begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix} \begin{bmatrix} F_n & \mathbf{0} \\ \mathbf{0} & F_n \end{bmatrix} P_{2n} =$$

$$= \begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix} \begin{bmatrix} I_{n/2} & D_{n/2} & & \\ I_{n/2} & -D_{n/2} & \multicolumn{2}{c}{\mathbf{0}} \\ \multicolumn{2}{c}{\mathbf{0}} & I_{n/2} & D_{n/2} \\ & & I_{n/2} & -D_{n/2} \end{bmatrix} \begin{bmatrix} F_{n/2} & \mathbf{0} & & \\ \mathbf{0} & F_{n/2} & \multicolumn{2}{c}{\mathbf{0}} \\ \multicolumn{2}{c}{\mathbf{0}} & F_{n/2} & \mathbf{0} \\ & & \mathbf{0} & F_{n/2} \end{bmatrix} \begin{bmatrix} P_n & \mathbf{0} \\ \mathbf{0} & P_n \end{bmatrix} P_{2n}$$

- We started with $(2n)^2$ multiplications and manage to reduce them to $2 \times (n)^2 + n$ multiplications!!
- Now the $n^2$ multiplications are reduced to $2 \times (n/2)^2 + n/2$ multiplications!!

# The Fast Fourier Transform

- I can carry on this recursive procedure until I reach $1 \times 1$ Fourier matrices.
- I will have a large number of matrices piling up.
- It can be proven that if we start with a matrix of size $n^2$ the total number of multiplications is reduced to

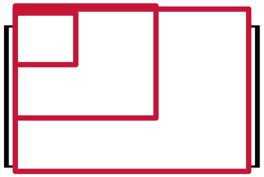
$$\frac{1}{2}n\log_2(n)$$

- Consider $n = 1024 = 2^{10}$. In that case $n^2 > 1{,}000{,}000$.
- $\frac{1}{2}1024\log_2(1024) = 5 \times 1024$.
- We reduced the multiplications from $1024 \times 1024$ to $5 \times 1024$, i.e., by a factor of $200$!
- The Fast Fourier Transform is one of the most important algorithms in modern scientific computing!

# Positive Definite Matrix

- We are talking about symmetric matrices.
- We have various tests. We take the 2 x 2 case $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$
  - » Eigenvalues are positive $\lambda_1 > 0, \lambda_2 > 0$
  - » All determinates of leading ("north west") sub-matrices are positive

$$a > 0, ac - b^2 > 0$$



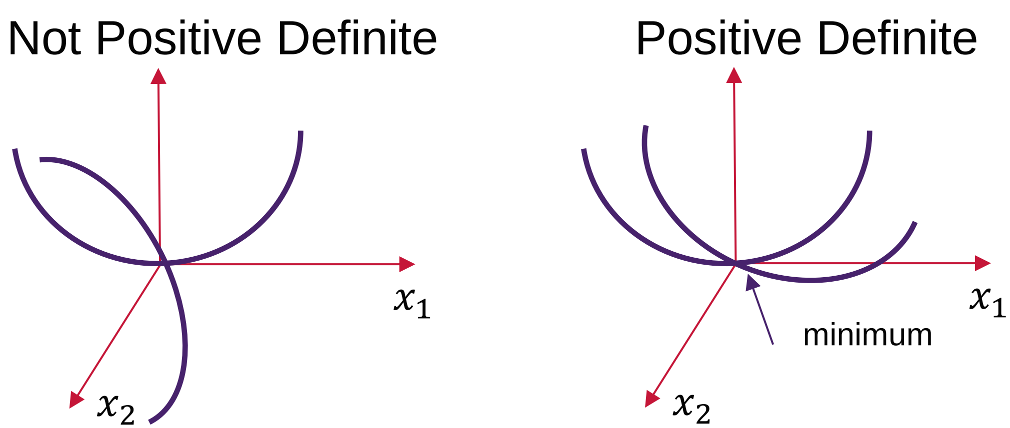
  - » Pivots are positive $a > 0$, $\frac{ac-b^2}{a} > 0$
  - » Quadratic form is positive $x^T A\, x > 0$, $x$ is any vector!

# Positive Definite Matrix

- Example: Take the matrix $\begin{bmatrix} 2 & 6 \\ 6 & x \end{bmatrix}$
  - » Which sufficiently large values of $x$ makes the matrix positive definite? The answer is $x > 18$. In that case we obtain the matrix $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$.
  - » For $x = 18$ the matrix is positive semi-definite. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 20$. **One of its eigenvalues is zero.**
  - » It has only one pivot since the matrix is singular. The pivots are $2$ and $0$. **The pivot test doesn't quite pass!**
  - » Its quadratic form is $[x_1 \quad x_2] \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 18x_2^2$.
    This is equal to $ax_1^2 + 2bx_1x_2 + cx_2^2 > 0$.
    In our case the matrix marginally failed the test.

# Graph of Quadratic Form

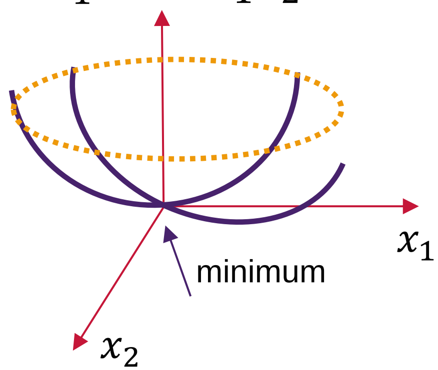
- Quadratic form $f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ : Let's look at the graph.




- For the positive definite case we have:
  - » First derivatives are zero. This condition is not enough!
  - » Second derivatives' matrix is positive definite $\begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix}$, $f_{x_1x_1}f_{x_2x_2} - 2f_{x_1x_2} > 0$.
  - » Positive for a number turns into positive-definite for a matrix!

# Graph of Quadratic Form

- Example: $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$, $trace(A) = 22 = \lambda_1 + \lambda_2$, $det(A) = 4 = \lambda_1 \lambda_2$
- $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 20x_2^2$
  $f(x_1, x_1) = 2x_1^2 + 12x_1x_2 + 20x_2^2 = 2(x_1 + 3x_2)^2 + 2x_2^2$




- A horizontal intersection could be $f(x_1, x_1) = 1$. It is an ellipse.
- Its quadratic form is $2(x_1 + 3x_2)^2 + 2x_2^2 = 1$.

# Graph of Quadratic Form

- Example: $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$, $trace(A) = 22 = \lambda_1 + \lambda_2$, $det(A) = 4 = \lambda_1 \lambda_2$

- $f(x_1, x_1) = 2x_1^2 + 12x_1x_2 + 20x_2^2 = \mathbf{2}(x_1 + \mathbf{3}x_2)^2 + \mathbf{2}x_2^2$

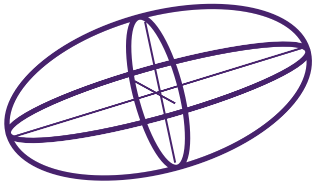
- Note that computing the square form is effectively elimination

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow[(2)-3(1)]{} \begin{bmatrix} \mathbf{2} & 6 \\ 0 & \mathbf{2} \end{bmatrix} = u \text{ and } L = \begin{bmatrix} 1 & 0 \\ \mathbf{3} & 1 \end{bmatrix}$$

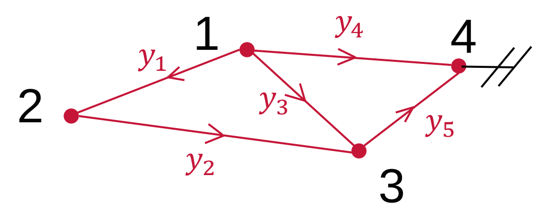
- The pivots and the multipliers appear in the quadratic form when we compute the square
- Pivots are the square multipliers so positive pivots imply sum of squares and hence positive definiteness

## Graph of Quadratic Form

- Example: Consider the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$
- The leading ("north west!) determinants are 2,3,4.
- The pivots are 2, 3/2, 4/3.
- The quadratic form is $\boldsymbol{x}^T A\, \boldsymbol{x} = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$.
- The eigenvalues of *A* are $\lambda_1 = 2 - \sqrt{2}, \lambda_2 = 2, \lambda_3 = 2 + \sqrt{2}$
- The matrix *A* is positive definite when $\boldsymbol{x}^T A\, \boldsymbol{x} > 0$. ***This matrix is p.d!***
- The intersection of the 4 dimensional "parabola" $\boldsymbol{x}^T A\, \boldsymbol{x} = 1$ is an
  - ➢ Ellipsoid with "principal" axes in the direction of eigenvectors.
  - ➢ The length of the axes is determined by the eigenvalues.



# Remember Graphs and Networks




- Summarizing all the equation

  - Potential differences $e = Ax$
  - Ohm's Law $y = Ce$
  - Kirchoff's Current Law $A^T y = 0$

- The above equations can be written in a single basic equation $A^T CAx = 0$