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# Maths for Signals and Systems

## Linear Algebra in Engineering

### Lectures 16-17, Tuesday 15th November 2016

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In this set of lectures we will talk about two applications:

- Discrete Fourier Transforms
- An application of linear system theory: graphs and networks

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# The Discrete Fourier Transform (DFT) matrix

- The $n \times n$ Fourier matrix is defined as:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & \cdots & w^{(n-1)} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(n-1)} & w^{2(n-1)} & \cdots & w^{(n-1)(n-1)} \end{bmatrix}$$

- In this matrix we will number the first row and column with $0$.
- We define $w = e^{-\mathbf{i}\frac{2\pi}{n}}$. For $w$ is preferable to use polar representation.
- $F_n(i,j) = w^{ij}$.
- We must stress out that it is better to use the notation $w_n$ instead of $w$.
- I have, in general, avoided this notation to make things look simpler but occasionally I used it.

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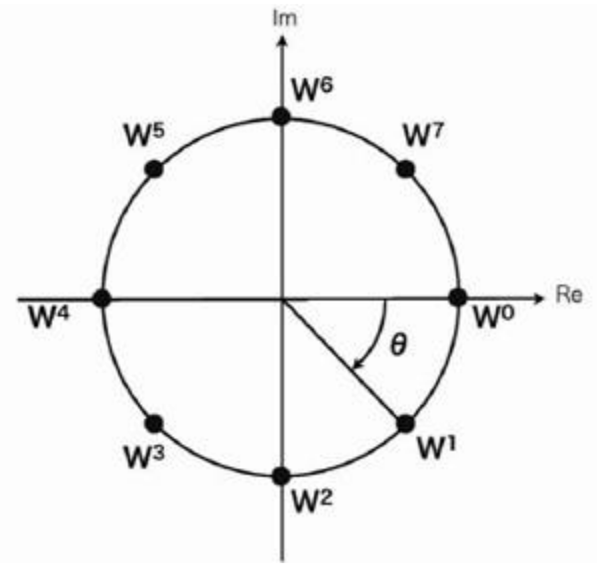
# The Discrete Fourier Transform (DFT) matrix cont.

- The parameter $w = e^{-\mathbf{i}\frac{2\pi}{n}}$ lies on the unit circle shown below. The case depicted below refers to $n = 8$ where the points $w^m = e^{-\mathbf{i}\frac{2\pi m}{8}}, m = 0, \dots, 7$ of the second row (row 1) of the Fourier matrix are shown.




- We must stress out that the Fourier matrix is totally constructed out of numbers of the form ${w_n}^k$.

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# The Discrete Fourier Transform (DFT) matrix for n=4

- The parameter $w_4 = e^{-\mathbf{i}\frac{2\pi}{4}} = e^{-\mathbf{i}\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + \mathbf{i}\sin\left(-\frac{\pi}{2}\right) = -\mathbf{i}$.
- The quantities inside Fourier matrix are $1, \mathbf{i}, , \mathbf{i}^2, \mathbf{i}^3, \mathbf{i}^4, \mathbf{i}^6, \mathbf{i}^9$.
- $, i^2$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathbf{i} & (-\mathbf{i})^2 & (-\mathbf{i})^3 \\ 1 & (-\mathbf{i})^2 & (-\mathbf{i})^4 & (-\mathbf{i})^6 \\ 1 & (-\mathbf{i})^3 & (-\mathbf{i})^6 & (-\mathbf{i})^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathbf{i} & (-\mathbf{i})^2 & (-\mathbf{i})^3 \\ 1 & (-\mathbf{i})^2 & (-\mathbf{i})^0 & (-\mathbf{i})^2 \\ 1 & (-\mathbf{i})^3 & (-\mathbf{i})^2 & (-\mathbf{i})^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathbf{i} & -1 & \mathbf{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathbf{i} & -1 & -\mathbf{i} \end{bmatrix}$$

- The columns of this matrix are orthogonal.
- Remember that the inner product of 2 complex vectors is $y^{*T}x = y^H x$.

# The Discrete Fourier Transform (DFT) matrix for n=4 cont.

- I can show that the columns are orthogonal but they are not orthonormal.
- I can fix this by dividing the Fourier matrix with the length of the rows (columns). In the case of $n = 4$ it is 2. Therefore, I can write:

$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathbf{i} & -1 & \mathbf{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathbf{i} & -1 & -\mathbf{i} \end{bmatrix}$$

- We can easily show that ${F_4}^H F_4 = I$.

# The Fast Fourier Transform (FFT)

- It can be proven that there is a connection between $F_{2n}$ and $F_n$.
- This is expected from the fact that ${w_{2n}}^2 = e^{-\mathbf{i}\frac{4\pi}{2n}} = e^{-\mathbf{i}\frac{2\pi}{n}} = w_n$. It can be shown that:

diagonal matrix $D_n$

$$[F_{2n}] = \begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix} \begin{bmatrix} F_n & \mathbf{0}_n \\ \mathbf{0}_n & F_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

permutation matrix $P_{2n}$

- When $[F_{2n}]$ is multiplied by a column vector in order to obtain the Fourier Transform of the signal, we require $(2n)^2$ multiplications.
- When $[F_{2n}]$ is decomposed as above, $P_{2n}$ does not contribute to multiplications, $\begin{bmatrix} F_n & \mathbf{0}_n \\ \mathbf{0}_n & F_n \end{bmatrix}$ requires $2 \times (n)^2$ multiplications and $\begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix}$ requires $n$ multiplications.
- In total $2 \times (n)^2 + n < (2n)^2$.

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# The Fast Fourier Transform (FFT) cont.

- In the previous analysis the matrix $D_n$ is defined as:

$$D_n = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{n-1} \end{bmatrix}$$

- We start requiring $(2n)^2$ multiplications and manage to reduce them to $2 \times (n)^2 + n$ multiplications.

# The Fast Fourier Transform (FFT) cont.

- The next step is to break the $F_n$ down. We use the above idea recursively.

$$[F_{2n}] = \begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix} \begin{bmatrix} F_n & \mathbf{O}_n \\ \mathbf{O}_n & F_n \end{bmatrix} P_{2n} =$$

$$= \begin{bmatrix} I_n & D_n \\ I_n & -D_n \end{bmatrix} \begin{bmatrix} \begin{matrix} I_{n/2} & D_{n/2} \\ I_{n/2} & -D_{n/2} \end{matrix} & \mathbf{O}_n \\ \mathbf{O}_n & \begin{matrix} I_{n/2} & D_{n/2} \\ I_{n/2} & -D_{n/2} \end{matrix} \end{bmatrix} \begin{bmatrix} \begin{matrix} F_{n/2} & \mathbf{O}_{n/2} \\ \mathbf{O}_{n/2} & F_{n/2} \end{matrix} & \mathbf{O}_n \\ \mathbf{O}_n & \begin{matrix} F_{n/2} & \mathbf{O}_{n/2} \\ \mathbf{O}_{n/2} & F_{n/2} \end{matrix} \end{bmatrix} \begin{bmatrix} P_n & \mathbf{O}_n \\ \mathbf{O}_n & P_n \end{bmatrix} P_{2n}$$

- We started with $(2n)^2$ multiplications and manage to reduce them to $2 \times (n)^2 + n$ multiplication.
- Now the $n^2$ multiplications are reduced to $2 \times (n/2)^2 + n/2$ multiplications.

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# The Fast Fourier Transform (FFT) cont.

- We can carry on this recursive procedure until we reach $1 \times 1$ Fourier matrices.
- We will have a large number of matrices piling up.
- It can be proven that if we start with a matrix of size $n \times n$ the total number of multiplications is reduced to

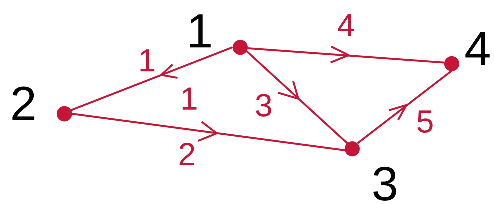
$$\frac{1}{2} n \log_2(n)$$

- Consider $n = 1024 = 2^{10}$. In that case $n^2 > 1{,}000{,}000$.
- $\frac{1}{2} 1024 \log_2(1024) = 5 \times 1024$.
- We reduced the multiplications from $1024 \times 1024$ to $5 \times 1024$, i.e., by a factor of $200$.
- **The Fast Fourier Transform is one of the most important algorithms in modern scientific computing.**

# Directed graphs and networks: The incidence matrix

- A **graph** is a mathematical model which consists of a set of **nodes** and **edges** denoted as:

  Graph={nodes, edges}



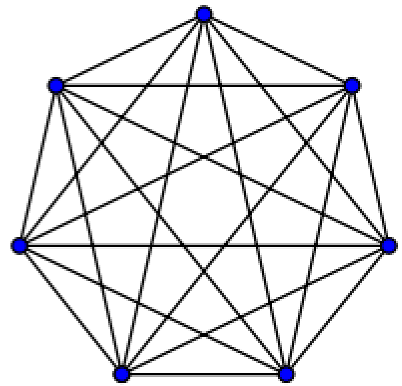

- Graphs are used in various applications.
- A graph can be represented by a matrix called **incidence matrix**.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

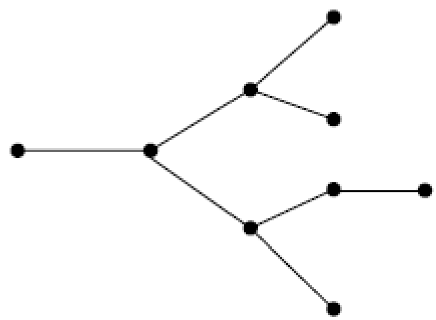
- Each row corresponds to an edge. Row numbers are edge numbers.
- Each column corresponds to a node. Column numbers are node numbers.
- The element $A_{ij} = 1$ if an arrow points towards node $j$ accross edge $i$.
- The element $A_{ij} = -1$ if an arrow leaves node $j$ accross edge $i$.

# Types of graphs

- A graph where every pair of nodes is connected with an edge is a **complete graph.** It has the maximum number of edges $m = \frac{1}{2}n(n-1)$ where $m$ is the number of edges and $n$ is the number of nodes.

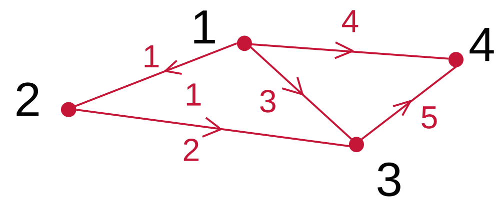


- A graph without closed loops is a **tree**. It has the minimum number of edges $m = n - 1$.



# Directed graphs and networks: The incidence matrix.

- The graph given previously is neither a complete graph nor a tree.




- Let us focus again on the **incidence matrix**.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- Observe that Row 3 = Row 1 + Row 2.
  - We also observe that edges 1, 2 and 3 form a closed loop.
  - We can make the statement that closed loops correspond to dependent rows.
  - Independent rows come from trees.
  - $\text{rank}(A) = 3$. Independent rows are 1, 2, 4 or 1, 2, 5 or 1, 4, 5 or 2, 4, 5.
  - Furthermore, $\text{rank}(A) = 3$ tells us that after 3 edges we start forming loops.

# Graphs and networks. Potential differences Ax

- Let us find the null space of the matrix that corresponds to the graph of interest:

$$Ax = \mathbf{0} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \mathbf{0}$$

- In a real life circuit $Ax$ is a vector of potential or voltage differences.
- If the graph represents and electronic circuit, the elements of vector $x$ may represent potentials at nodes (e.g. voltages).
- $x_i - x_j$ represents the difference in potential across certain edges.
- We see that a solution of the above system is $x = [1 \quad 1 \quad 1 \quad 1]^T$.
- The null space is formed by vectors $c[1 \quad 1 \quad 1 \quad 1]^T$ and $\dim(N(A)) = 1$ .
- The solution to the above system is obtained subject to a scalar $c$.
- Since $n = 4$ and and $\dim(N(A)) = 1$, we see again that $\mathrm{rank}(A) = 3$ .
- By fixing the potential at node one to 0 we remove the first column and we solve for the remaining potentials. In Electrical Engineering this is translated as node 1 been “grounded”.

# Graphs and networks: Kirchoff's Current Law (KCL)

- Let us consider the equation

$$A^T y = 0 \Rightarrow \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \mathbf{0}$$

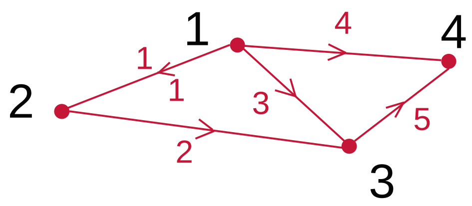
- The vector $y$ represents currents across the edges (or it could represent a force).
- The equation $A^T y = 0$ represents Kirchoff's Current Law (KCL):

  **KCL: Current that flows in is equal to current that flows out at each node.**
- Note that there is a matrix $C$ that connects currents and potential differences at the edges, and represent Ohm's law: $y = Ce$. We will talk about this later.

# Graphs and networks: Kirchoff's Current Law (KCL) cont.

- The equation $A^T y = 0$ is Kirchoff's Current Law.

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \mathbf{0}$$



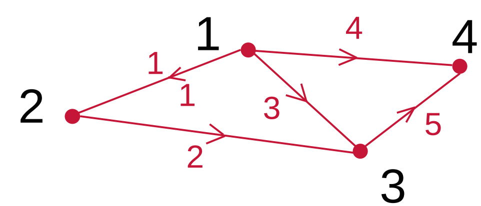

- The first equation refers to node one and indicates that the net current flow is zero. Similarly we get:

$$-y_1 - y_3 - y_4 = 0$$
$$y_1 - y_2 = 0$$
$$y_2 + y_3 - y_5 = 0$$
$$y_4 + y_5 = 0$$

# Graphs and networks: Kirchoff's Current Law cont.

- The three solution vectors below, that satisfy Kirchoff's Current Law, represent total current running across the three possible loops.

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$




- We can see that the third solution (current running across the big external loop 3) is not independent from the first two solutions.
- The null space of $A^T$ is, therefore, two dimensional, which is the same as the number of **small** loops. This is expectable also from the fact that:

$$\dim(N(A^T)) = m - r = 5 - 3 = 2$$

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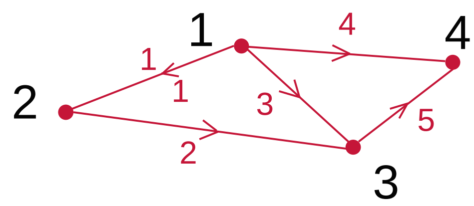
# Graphs and networks: row space of incidence matrix

- Consider the columns space of $A^T$ which is the row space of $A$.

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$




- The pivot columns of $A^T$ are located on the first, second and the fourth columns. These are associated with edges that form a graph without loops. As mentioned this graph is called a **tree**.
- The following relationships hold:
  - $\dim(N(A^T)) = m - r \Rightarrow \#\text{small loops} = \#\text{edges} - (\#\text{nodes} - 1)$
  - $\#\text{nodes} - \#\text{edges} + \#\text{small loops} = 1$. This relationship is known as **Euler's formula**.

# Real networks: Ohm's Law

- In real life networks we have: current=$c$ ·potential differences.
- $c$ is the so called **conductance**.
- It tells us how easily flow gets through an edge (high for metal, low for plastic etc.)
- Current through an edge is a function of the conductance across this edge **only** and the potential difference across the edge.
- Therefore, each edge is associated with a conductance.
- This yields $y = Ce$, with $C$ being a diagonal matrix.
- The relationship $y = Ce$, is the so called **Ohm's Law**.

# Summary

- In real life networks we have:
  - Potential differences: $e = Ax$
  - Ohm's Law: $y = Ce$
  - Kirchoff's Current Law: $A^T y = \mathbf{0}$

- The above three equations can be merged in a single equation as follows:

$$A^T y = \mathbf{0} \Rightarrow A^T Ce = \mathbf{0} \Rightarrow A^T CAx = \mathbf{0}$$