Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

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Eigenvectors and eigenvalues

- Consider a matrix A and a vector x.
- The operation Ax produces a vector y at some direction.
- I am interested in vectors y which lie in the same direction as x.
- In that case I have $Ax = \lambda x.$
- When the above relationship holds, x is called an eigenvector and λ is called an eigenvalue of matrix A.
- If A is singular then $\lambda = 0$ is an eigenvalue.
- **Problem:** How do we find the eigenvectors and eigenvalues of a matrix?

Eigenvectors and eigenvalues of a projection matrix

- **Problem:** What are the eigenvectors x' and eigenvalues λ' of a projection matrix P?

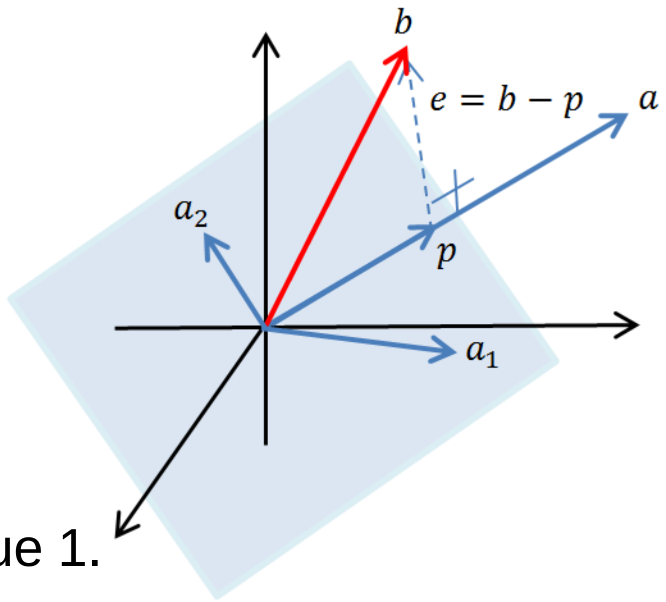
  In the figure, consider the matrix P which projects vector b onto vector p.

  Question: Is b an eigenvector of P?

  Answer: NO, because b and Pb lie in different directions!

  Question: What vectors are eigenvectors of P?

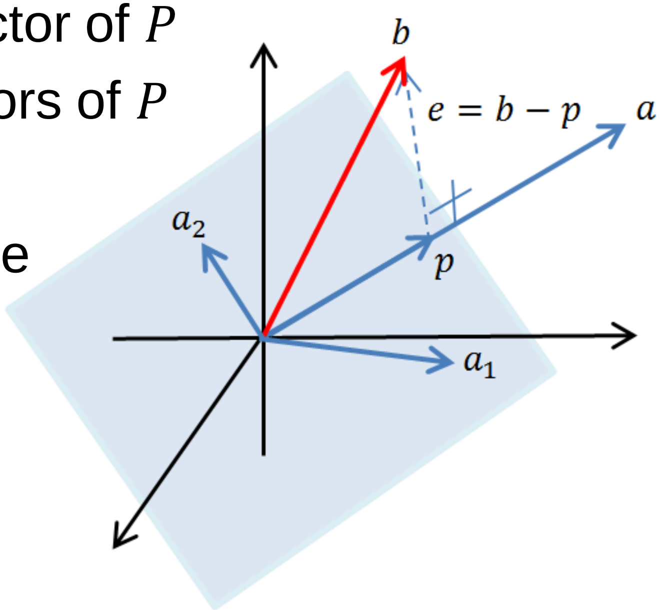
  Answer: Vectors x which lie on the projection plane already! In that case $Px = x$ and therefore x is an eigenvector with eigenvalue 1.



Eigenvectors and eigenvalues of a projection matrix (cont.)

- The eigenvectors of P are vectors x which lie on the projection plane already! In that case $Px = x$ and therefore x is an eigenvector with eigenvalue 1.
- We can find 2 independent eigenvectors of P which lie on the projection plane.
- **Problem:** In the 3D space we can find 3 independent vectors. Can you find a third eigenvector of P that is perpendicular to the eigenvectors of P that lie on the projection plane?

  Answer: YES! Any vector e perpendicular to the plane. In that case $Pe = 0e = 0$.
- The eigenvalues of P are $\lambda = 0$ and $\lambda = 1$.



Eigenvectors and eigenvalues of a permutation matrix

- Consider the permutation matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- **Problem:** Can you give an eigenvector of the above matrix? Or can you think of a vector that if permuted is still a multiple of itself?

Answer: YES! It is the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the corresponding eigenvalue is $\lambda = 1$. And furthermore, the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = -1$.

- $n \times n$ matrices will have n eigenvalues.
- It is not so easy to find them!
- But there is an amazing fact! The sum of the eigenvalues, called the **trace** of a matrix, equals the sum of the diagonal elements of the matrix.
- Therefore, in the previous example, once I found an eigenvalue $\lambda = 1$, I should suspect that there is another eigenvalue $\lambda = -1$.

Problem: Solve $Ax = \lambda x$

- $Ax = \lambda x \Rightarrow Ax - \lambda x = 0$ (0 is the zero vector).
  $(A - \lambda I)x = 0$

  In order for the above set of equations to have a non-zero solution, the matrix $(A - \lambda I)$ must be singular. Therefore, $det(A - \lambda I) = 0$.
- I now have an equation for λ. It is called the **characteristic equation**, or the eigenvalue equation. The idea then is to find λs first.
- I might have repeated λs. **These mean trouble but I will deal with this later!**
- After I find λ, I can find x from $(A - \lambda I)x = 0$. Basically, I will be looking for the null space of $(A - \lambda I)$.

Solve $Ax = \lambda x$. An example

- Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. A symmetric matrix has real eigenvalues!
- Eigenvectors of a symmetric matrix can be chosen to be orthogonal.
- $det(A - \lambda I) = (3-\lambda)^2 - 1 = 0 \Rightarrow 3 - \lambda = \pm 1 \Rightarrow \lambda = 3 \pm 1 \Rightarrow \lambda_1 = 4$, $\lambda_2 = 2$.
  Or $det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = 0$. Note that $6 = \lambda_1 + \lambda_2$ and $8 = det(A) = \lambda_1 \lambda_2$.
- Find the eigenvector for $\lambda_1 = 4$.
  $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = y$
- Find the eigenvector for $\lambda_2 = 2$.
  $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y$
- There are entire lines of eigenvectors, not single eigenvectors!

Compare the two problems.

- Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ with eigenvectors $\begin{bmatrix} x \\ x \end{bmatrix}$ and $\begin{bmatrix} -x \\ x \end{bmatrix}$ and eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$.

- Consider the matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with eigenvectors $\begin{bmatrix} x \\ x \end{bmatrix}$ and $\begin{bmatrix} -x \\ x \end{bmatrix}$ and eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

- We observe that $A = B + 3I$. The eigenvalues of A are obtained from the eigenvalues of B if we increase them by 3!

- The eigenvectors of A and B are the same!

Generalization of the above observation

- Consider the matrix $A = B + cI$.
- Consider an eigenvector x of B with eigenvalue λ. Then:

  $Bx = \lambda x$ and therefore,

  $Ax = (B + cI)x = Bx + cIx = Bx + cx = \lambda x + cx = (\lambda + c)x$

  A has the same eigenvectors with B with eigenvalues $\lambda + c$.

- BUT: There isn't any property that enables us to find the eigenvalues of $A + B$ and AB.

Example

- Take a matrix that rotates every vector by 90^{o}.
- This is $Q = \begin{bmatrix} \cos(90) & -\sin(90) \\ \sin(90) & \cos(90) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- $\lambda_1 + \lambda_2 = 0$ and $det(Q) = \lambda_1 \lambda_2 = 1$
- What vector can be parallel to itself after rotation?
- $det(Q - \lambda I) = det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$
- In that case we have an anti-symmetric matrix with $Q^T = Q^{-1} = -Q$.
- The eigenvalues come in complex conjugate pairs.

Example

- Consider $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$
- $\lambda_1 + \lambda_2 = 6$ and $det(\lambda_1 \lambda_2) = 9$
- $det(A - \lambda I) = det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 = 0 \Rightarrow \lambda = 3$
- The eigenvalues of a triangular matrix are the values of the diagonal.
- In that case we have

  $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \Rightarrow y = 0$ and x can be any number.
- In that case of repeated eigenvalues, we don't have 2 independent eigenvectors!

Matrix diagonalization for the case of independent eigenvectors

- Suppose we have n independent eigenvectors of a matrix A. We call them x_i.
- We put them in the columns of a matrix S.
- We form the matrix $AS = A[x_1 \quad x_2 \quad \cdots \quad x_n] = [\lambda_1 x_1 \quad \lambda_2 x_2 \quad \ldots \quad \lambda_n x_n] =$

$$[x_1 \quad x_2 \quad \cdots \quad x_n]\begin{bmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_n\end{bmatrix} = S\Lambda \Rightarrow AS = S\Lambda$$

$S^{-1}AS = \Lambda$ or $A = S\Lambda S^{-1}$

Matrix diagonalization: Eigenvalues of A^k

- If $Ax = \lambda x \Rightarrow A^2x = \lambda Ax \Rightarrow A^2x = \lambda^2 x$.
- Therefore, the eigenvalues of A^2 are λ^2.
- The eigenvectors of A^2 remain the same.
- $A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$
- $A^k = S\Lambda^k S^{-1}$
- $\lim(A^k) = 0$ if the eigenvalues of A have the property $|\lambda_i| < 1$
- A matrix has n independent eigenvectors and therefore is diagonalizable if all the eigenvalues are different (non repeated eigenvalues exist).
- If I have repeated eigenvalues I may, or may not have independent eigenvectors (consider the identity matrix!)
- Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

Application at last! A first order system which evolves with time

- The system follows an equation of the form $u_{k+1} = Au_k$.
- u_k is the vector which consists of the system parameters which evolve with time.
- The eigenvalues of A characterize fully the behavior of the system.
- I start with a given vector u_0.

  $u_1 = Au_0$, $u_2 = A^2u_0$ and in general $u_k = A^k u_0$
- To really solve the above, I write $u_0 = c_1x_1 + c_2 x_2 + \cdots + c_nx_n$ where x_i are the eigenvectors of matrix A.
- $Au_0 = c_1Ax_1 + c_2 Ax_2 + \cdots + c_nAx_n = c_1\lambda_1x_1 + c_2 \lambda_2x_2 + \cdots + c_n\lambda_nx_n$

  $A^{100}u_0 = c_1{\lambda_1}^{100}x_1 + c_2 {\lambda_2}^{100}x_2 + \cdots + c_n{\lambda_n}^{100}x_n = S\Lambda^{100}c$

  c is a column vector that contains the coefficients c_i.

Fibonacci example. Convert a second order scalar problem into a first order system

- I will take two numbers which I call $F_0 = 0$ and $F_1 = 1$.
- The Fibonacci sequence of numbers is given by the two initial numbers given above and the relationship $F_k = F_{k-1} + F_{k-2}$.
- 0,1,1,2,3,5,8,13 and so on.
- How can I get a formula for the 100th Fibonacci number?
- Here is the trick:

  I define a vector $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ and an extra equation $F_{k+1} = F_{k+1}$

- $u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Au_k$

Fibonacci example. Convert a second order scalar problem into a first order system

- The eigenvalues of A are obtained from

$$det\begin{bmatrix}1-\lambda & 1\\ 1 & -\lambda\end{bmatrix} = -(1-\lambda)\lambda - 1 = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$$

- Observe the analogy between $\lambda^2 - \lambda - 1 = 0$ and $F_k - F_{k-1} - F_{k-2} = 0$.
- $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$. Eigenvalues add up to 1. The matrix A is diagonalizable.
- How can I get a formula for the 100th Fibonacci number?
- $F_{100} \approx c_1(\frac{1+\sqrt{5}}{2})^{100}$. The term $c_2(\frac{1-\sqrt{5}}{2})^{100}$ becomes negligible.
- The eigenvectors are $x_1 = \begin{bmatrix}\lambda_1\\ 1\end{bmatrix}$, and $x_2 = \begin{bmatrix}\lambda_2\\ 1\end{bmatrix}$.
- $u_0 = \begin{bmatrix}F_1\\ F_2\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = c_1\begin{bmatrix}\lambda_1\\ 1\end{bmatrix} + c_2\begin{bmatrix}\lambda_2\\ 1\end{bmatrix}$. From this equation we find c_1 and c_2.

More applications: First order differential equations $\frac{du}{dt} = Au$

- **Problem:** Solve the system of differential equations:

$$\frac{du_1}{dt} = -u_1 + 2u_2$$
$$\frac{du_2}{dt} = u_1 - 2u_2$$

  We set $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and we see that the system's matrix is $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$.

  The matrix A is singular. One of the eigenvalues is zero.

  Therefore, the eigenvalues are $\lambda_1 = 0,\ \lambda_2 = -3!$

- The solution of the above system depends exclusively on the eigenvalues of A.
- The eigenvectors are $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

First order differential equations $\frac{du}{dt} = Au$

- **Solution:** $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$
- **Problem:** Verify the above by plugging-in $e^{\lambda_i t} x_i$ to the equation $\frac{du}{dt} = Au$.
- Let's find $u(t) = c_1 e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- c_1, c_2 comes from the initial conditions.
- $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = \frac{1}{3}, c_2 = 1/3.$
- **Steady state** of the system is $u(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.
- **Stability** is achieved if the real part of the eigenvalues is negative.
- Note that the complex eigenvalues appear in conjugate pairs.

First order differential equations $\frac{du}{dt} = Au$

- For $t = 0$, the relationship $u(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ becomes

$$u(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- The above can be written in matrix form as:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow Sc = u(0)$$

- $c_1 = \frac{1}{3}, c_2 = 1/3$.

Stability: First order differential equations $\frac{du}{dt} = Au$

- **Stability** is achieved if the real part of the eigenvalues is negative.
- We <u>**do**</u> have a steady state if at least one eigenvalue is 0 and the rest of the eigenvalues have negative real part.
- We blow up if at least one eigenvalue has a positive real part!
- For stability the trace of the system's matrix must be negative.
- A negative trace, though, does not guarantee stability **(why?)**
- A negative trace and positive determinant does guarantee stability! (I am sure you know why!)

Back to the equation $\frac{du}{dt} = Au$

- I set $u = Sv$ and therefore the differential equation becomes:
- $S\frac{dv}{dt} = ASv \Rightarrow \frac{dv}{dt} = S^{-1}ASv = \Lambda v$
- This is an amazing result!!!
- I start from a system of equations $\frac{du}{dt} = Au$ which are **coupled** (or "dependent" or "correlated") and I end up with a set of equations which are **decoupled** and easier to solve!!!
- I am hoping to get at some point:
  $v(t) = e^{\Lambda t}v(0)$ and $u(t) = Se^{\Lambda t}S^{-1}u(0)$ with $e^{At} = Se^{\Lambda t}S^{-1}$
- **Question:** What is the exponential of a matrix?

Matrix exponentials e^{At}

- Taylor series $e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$
- Note that $e^x = \sum_0^\infty \frac{x^n}{n!}$. The exponential series always converges!
- Furthermore, note that $\frac{1}{1-x} = \sum_0^\infty x^n$.

  For matrices we have $(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots$

  This sum converges if $|\lambda(At)| < 1$.
- The function that I am chiefly interested in is e^{At} and I would like to connect it to S and Λ.
- $e^{At} = I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2} + \frac{S\Lambda^3 S^{-1}t^3}{6} + \cdots + \frac{S\Lambda^n S^{-1}t^n}{n!} + \cdots = Se^{\Lambda t}S^{-1}$
- **Question:** What assumption is built-in to this formula, that is not built to the original formula in the first line?

  Answer: The assumption is that A must be diagonalizable.

Diagonal matrix exponentials $e^{\Lambda t}$

- The exponential $e^{\Lambda t}$ of a diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \text{ is}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

- As we already showed

$$\lim_{t \to \infty} e^{\Lambda t} = 0 \text{ if } \mathrm{Re}(\lambda_i) < 0, \forall i$$

$$\lim_{t \to \infty} \Lambda^t = 0 \text{ if } |\lambda_i| < 0, \forall i$$

Second order differential equations

- How do I change the second order differential equation

$$y'' + by' + ky = 0$$

  to two first order ones?

- I define $u = \begin{bmatrix} y' \\ y \end{bmatrix}$ and therefore,

$$u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} u$$

Higher order differential equations

- How do I change the n^{th} order differential equation

$$y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y = 0$$

  to n first order ones?

- I define $u = \begin{bmatrix} y^{(n-1)} \\ \vdots \\ y' \\ y \end{bmatrix}$ and therefore,

$$u' = \begin{bmatrix} y^{(n)} \\ y^{(n-1)} \\ \vdots \\ y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b_1 & -b_2 & \dots & -b_{n-2} & -b_{n-1} \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} u$$