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Maths for Signals and Systems

Linear Algebra in Engineering

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Eigenvectors and eigenvalues

- Consider a matrix A and a vector x.
- The operation Ax produces a vector y at some direction.
- I am interested in vectors y which lie in the same direction as x.
- In that case I have $Ax = \lambda x.$
- When the above relationship holds, x is called an **eigenvector** and λ is called an **eigenvalue** of matrix A.
- If A is singular then $\lambda = 0$ is an eigenvalue.
- **Problem:** How do we find the eigenvectors and eigenvalues of a matrix?

Eigenvectors and eigenvalues of a projection matrix

- **Problem:** What are the eigenvectors x' and eigenvalues λ' of a projection matrix P? In the figure, consider the matrix P which projects vector b onto vector p.

Question: Is b an eigenvector of P?

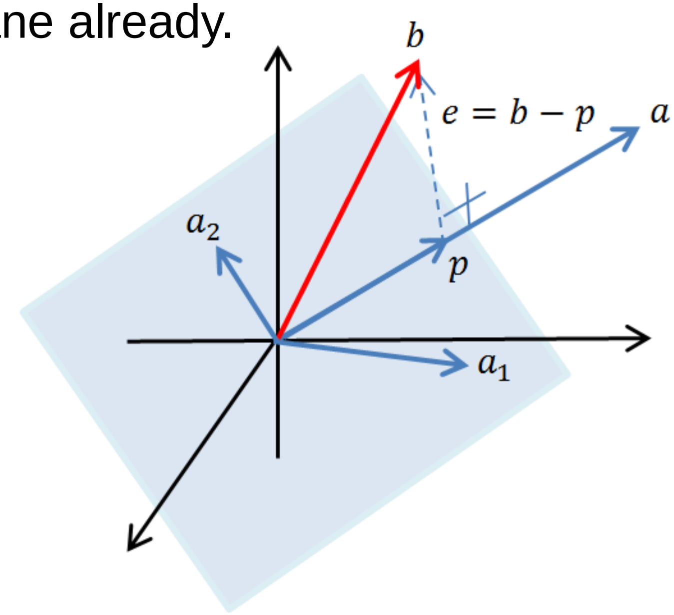
Answer: No, because b and Pb lie in different directions.

Question: What vectors are eigenvectors of P?

Answer: Vectors x which lie on the projection plane already.

In that case $Px = x$ and therefore

x is an eigenvector with eigenvalue 1.



Eigenvectors and eigenvalues of a projection matrix (cont.)

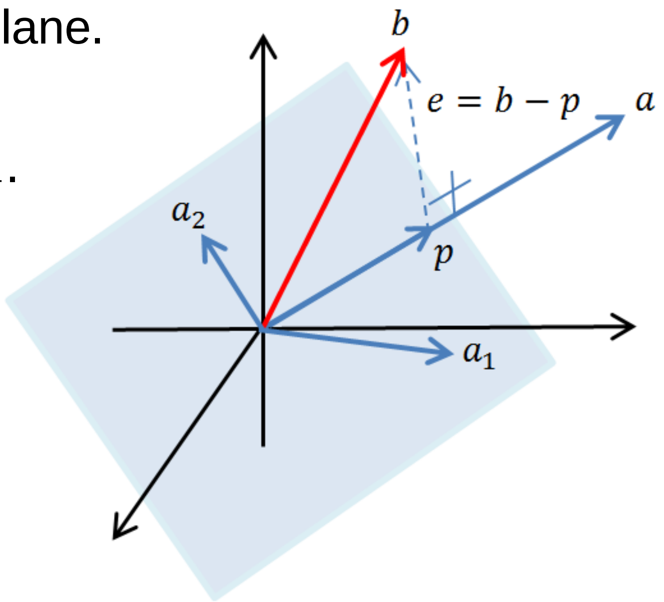
- The eigenvectors of P are vectors x which lie on the projection plane already. In that case $Px = x$ and therefore x is an eigenvector with eigenvalue 1.
- We can find 2 independent eigenvectors of P which lie on the projection plane.

Problem: In the 3D space we can find 3 independent vectors. Can you find a third eigenvector of P that is perpendicular to the eigenvectors of P that lie on the projection plane?

Answer: YES! Any vector e perpendicular to the plane.

In that case $Pe = 0e = 0$. Therefore,

the eigenvalues of P are $\lambda = 0$ and $\lambda = 1$.



Eigenvectors and eigenvalues of a permutation matrix

- Consider the permutation matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

  Problem: Can you give an eigenvector of the above matrix? Or can you think of a vector that if permuted is still a multiple of itself?

  Answer: YES. It is the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the corresponding eigenvalue is $\lambda = 1$.

  And furthermore, the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = -1$.

- $n \times n$ matrices will have n eigenvalues.
- It is not so easy to find them.
- **The sum of the eigenvalues, called the <u>trace</u> of a matrix, equals the sum of the diagonal elements of the matrix.**
- **The product of the eigenvalues equals the determinant of the matrix.**
- Therefore, in the previous example, once I found an eigenvalue $\lambda = 1$, I should suspect that there is another eigenvalue $\lambda = -1$.

Problem: Solve $Ax = \lambda x$

- Consider an eigenvector x of matrix A. In that case $Ax = \lambda x \Rightarrow Ax - \lambda x = 0$ (0 is the zero vector). Therefore, $(A - \lambda I)x = 0$.

  In order for the above set of equations to have a non-zero solution, the nullspace of $(A - \lambda I)$ must be non-zero, i.e., the matrix $(A - \lambda I)$ must be singular. Therefore, $\det(A - \lambda I) = 0$.

- I now have an equation for λ. It is called the **characteristic equation**, or the **eigenvalue equation**. Using this equations we can find the eigenvalues.

- I might have repeated λs. **This causes problems but I will deal with it later.**

- After I find λ, I can find x from $(A - \lambda I)x = 0$. Basically, I will be looking for the nullspace of $(A - \lambda I)$.

Solve $Ax = \lambda x$. An example.

- Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. **A symmetric matrix always has real eigenvalues.**
- Furthermore, **the eigenvectors of a symmetric matrix can be chosen to be orthogonal.**
- $\det(A - \lambda I) = (3-\lambda)^2 - 1 = 0 \Rightarrow 3 - \lambda = \pm 1 \Rightarrow \lambda = 3 \pm 1 \Rightarrow \lambda_1 = 4,\ \lambda_2 = 2.$
  Or $\det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = 0$. Note that $6 = \lambda_1 + \lambda_2$ and $8 = \det(A) = \lambda_1 \lambda_2$.
- Find the eigenvector for $\lambda_1 = 4$.
  $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = y$
- Find the eigenvector for $\lambda_2 = 2$.
  $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y$
- Notice that there are families of eigenvectors, not single eigenvectors.

Compare the two matrices given previously

- Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. As shown it has eigenvectors $\begin{bmatrix} x \\ x \end{bmatrix}$ and $\begin{bmatrix} -x \\ x \end{bmatrix}$ and eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$.

- Consider the matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, also with eigenvectors $\begin{bmatrix} x \\ x \end{bmatrix}$ and $\begin{bmatrix} -x \\ x \end{bmatrix}$ and eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

- **We observe that $\boldsymbol{A = B + 3I}$. The eigenvalues of $\boldsymbol{A}$ are obtained from the eigenvalues of $\boldsymbol{B}$ if we increase them by 3.**

- **The eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same.**

Generalization of the above observation

- Consider the matrix $A = B + cI$.
- Consider an eigenvector x of B with eigenvalue λ. Then $Bx = \lambda x$ and therefore, $Ax = (B + cI)x = Bx + cIx = Bx + cx = \lambda x + cx = (\lambda + c)x$
  A has the same eigenvectors with B with eigenvalues $\lambda + c$.
- **There aren't any properties that enable us to find the eigenvalues of $\boldsymbol{A + B}$ and $\boldsymbol{AB}$.**

Example

- Take a matrix that rotates every vector by 90^o.
- This is $Q = \begin{bmatrix} \cos(90) & -\sin(90) \\ \sin(90) & \cos(90) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- $\lambda_1 + \lambda_2 = 0$ and $\det(Q) = \lambda_1 \lambda_2 = 1$.
- **What vector can be parallel to itself after rotation?**
- $\det(Q - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$.
- In that case we have a **skew symmetric** (or **anti-symmetric**) matrix with
$$Q^T = Q^{-1} = -Q.$$
- The complex eigenvalues come in complex conjugate pairs if a matrix is real.

Example

- Consider $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.
- $\lambda_1 + \lambda_2 = 6$ and $\det(\lambda_1 \lambda_2) = 9$.
- $\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 = 0 \Rightarrow \lambda_{1,2} = 3$
- The eigenvalues of a triangular matrix are the values of the diagonal.
- For that particular case we have

  $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \Rightarrow y = 0$ and x can be any number.

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Matrix diagonalization
The case of independent eigenvectors

- Suppose we have n independent eigenvectors of a matrix A. We call them x_i.
- We put them in the columns of a matrix S.
- We form the matrix:

$$AS = A[x_1 \quad x_2 \quad \cdots \quad x_n] = [\lambda_1 x_1 \quad \lambda_2 x_2 \quad \dots \quad \lambda_n x_n] =$$

$$[x_1 \quad x_2 \quad \cdots \quad x_n]\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = S\Lambda \Rightarrow AS = S\Lambda$$

$S^{-1}AS = \Lambda$ or $A = S\Lambda S^{-1}$

Matrix diagonalization: Eigenvalues of A^k

- If $Ax = \lambda x \Rightarrow A^2 x = \lambda A x \Rightarrow A^2 x = \lambda^2 x$.
- Therefore, the eigenvalues of A^2 are λ^2.
- The eigenvectors of A^2 remain the same.
- $A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$
- $A^k = S\Lambda^k S^{-1}$
- $\lim(A^k) = 0$ if the eigenvalues of A have the property $|\lambda_i| < 1$.
- A matrix has n independent eigenvectors and therefore is diagonalizable if all the eigenvalues are different.
- If I have repeated eigenvalues I may, or may not have independent eigenvectors. As an example consider the identity matrix. Its eigenvectors are the row (or column) vectors. They are all independent. However, the eigenvalues are all equal to 1.
- Find the eigenvalues of $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, a \neq 0$. **This is a standard textbook matrix which cannot be diagonalized.**

Application
A first order system which evolves with time

- Consider a system that follows an equation of the form $u_{k+1} = Au_k$.
- u_k is the vector which consists of the system parameters which evolve with time.
- The eigenvalues of A characterize fully the behavior of the system.
- I start with a given vector u_0.

  $u_1 = Au_0$, $u_2 = A^2u_0$ and in general $u_k = A^k u_0$
- In order to solve this system I assume that I can write $u_0 = c_1x_1 + c_2\, x_2 + \cdots + c_nx_n$ where x_i are the eigenvectors of matrix A. This is a standard approach to the solution of this type of problem.
- $Au_0 = c_1Ax_1 + c_2\, Ax_2 + \cdots + c_nAx_n = c_1\lambda_1x_1 + c_2\, \lambda_2x_2 + \cdots + c_n\lambda_nx_n$

  $A^{100}u_0 = c_1{\lambda_1}^{100}x_1 + c_2\, {\lambda_2}^{100}x_2 + \cdots + c_n{\lambda_n}^{100}x_n = S\Lambda^{100}c$

  c is a column vector that contains the coefficients c_i.

  S, Λ are the matrices defined previously.

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Fibonacci example:
Convert a second order scalar problem into a first order system

- I will take two numbers which I call $F_0 = 0$ and $F_1 = 1$.
- The Fibonacci sequence of numbers is given by the two initial numbers given above and the relationship $F_k = F_{k-1} + F_{k-2}$.
- The sequence is 0,1,1,2,3,5,8,13 and so on.
- How can I get a formula for the 100th Fibonacci number?
- I will present a standard approach to this problem:

  I define a vector $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ and an extra equation $F_{k+1} = F_{k+1}$

- $u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Au_k$
- I managed to convert the second order scalar problem into a first order matrix problem.

Fibonacci example:
Convert a second order scalar problem into a first order system

- The eigenvalues of A are obtained from

$$\det\begin{bmatrix}1-\lambda & 1\\ 1 & -\lambda\end{bmatrix} = -(1-\lambda)\lambda - 1 = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$$

- Observe the analogy between $\lambda^2 - \lambda - 1 = 0$ and $F_k - F_{k-1} - F_{k-2} = 0$.
- $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$. Eigenvalues add up to 1. The matrix A is diagonalizable.
- It can be shown that the eigenvectors are $x_1 = \begin{bmatrix}\lambda_1\\ 1\end{bmatrix}$, and $x_2 = \begin{bmatrix}\lambda_2\\ 1\end{bmatrix}$.
- How can I get a formula for the 100th Fibonacci number?
- $u_{k+1} = \begin{bmatrix}F_{k+2}\\ F_{k+1}\end{bmatrix} = Au_k = A^2u_{k-1}$ etc. Therefore, $u_{100} = A^{100}u_0 = S\Lambda^{100}S^{-1}u_0$
- $\Lambda^{100} = \begin{bmatrix}{\lambda_1}^{100} & 0\\ 0 & {\lambda_2}^{100}\end{bmatrix} \cong \begin{bmatrix}{\lambda_1}^{100} & 0\\ 0 & 0\end{bmatrix}$, $S = \begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}$, $S^{-1} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{bmatrix}$
- $u_{100} \cong \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}{\lambda_1}^{100} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}{\lambda_1}^{100} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} =$
$\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}{\lambda_1}^{100}\\ 0\end{bmatrix} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}{\lambda_1}^{101}\\ {\lambda_1}^{100}\end{bmatrix} \Rightarrow F_{100} = \frac{1}{\lambda_1-\lambda_2}{\lambda_1}^{100}$

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Applications

First order differential equations $\frac{du}{dt} = Au$

Problem:

Solve the system of differential equations:

$$\frac{du_1(t)}{dt} = -u_1(t) + 2u_2(t), \qquad \frac{du_2(t)}{dt} = u_1(t) - 2u_2(t)$$

Solution:

- We convert the system of equations into a matrix form $\begin{bmatrix} \frac{du_1(t)}{dt} \\ \frac{du_2(t)}{dt} \end{bmatrix} = A \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$.
- We set $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$. We impose the **initial conditions** $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- The system's matrix is $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$. The matrix A is singular. One of the eigenvalues is zero. Therefore, the eigenvalues are $\lambda_1 = 0,\ \lambda_2 = -3$.
- The solution of the above system depends exclusively on the eigenvalues of A.
- The eigenvectors are $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Applications

First order differential equations $\frac{du}{dt} = Au$ cont.

- The solution of the above system of equations is of the form

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

Problem:

Verify the above by plugging-in $e^{\lambda_i t} x_i$ to the equation $\frac{du}{dt} = Au$.

- Let's find $u(t) = c_1 e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- c_1, c_2 comes from the initial conditions.

  $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow S \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = u(0) \Rightarrow c_1 = \frac{1}{3}, c_2 = \frac{1}{3}.$
- **The Steady State** of the system is defined as $u(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.
- **Stability** is achieved if the real part of the eigenvalues is negative.
- Note that if complex eigenvalues exists, they appear in conjugate pairs.

Stability: First order differential equations $\frac{du}{dt} = Au$

- **Stability** is achieved if the real part of the eigenvalues is negative.
- We do have a **steady state** if at least one eigenvalue is 0 and the rest of the eigenvalues have negative real part.
- The system is unstable if at least one eigenvalue has a positive real part.
- For stability the trace of the system's matrix must be negative.
- A negative trace, though, does not guarantee stability.

Stability: First order differential equations $\frac{du}{dt} = Au$

- Recall the matrix S which contains the eigenvectors of A defined previously.
- I set $u(t) = Sv(t)$ and therefore the differential equation becomes:
- $S\frac{dv(t)}{dt} = ASv(t) \Rightarrow \frac{dv(t)}{dt} = S^{-1}ASv(t) = \Lambda v(t)$
- This is an interesting result which can be found in various engineering applications.
- I start from a system of equations $\frac{du(t)}{dt} = Au(t)$ which are **coupled** (or **dependent** or **correlated**) and I end up with a set of equations which are **decoupled** and easier to solve.
- From the relationship $\frac{dv(t)}{dt} = \Lambda v(t)$ we get $v(t) = e^{\Lambda t}v(0)$ and $u(t) = Se^{\Lambda t}S^{-1}u(0)$ with $e^{At} = Se^{\Lambda t}S^{-1}$

Question: What is the exponential of a matrix? See next slide.

Matrix exponentials e^{At}

- Consider the Taylor series $e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$

  Note that $e^x = \sum_0^\infty \frac{x^n}{n!}$.
- Furthermore, note that $\frac{1}{1-x} = \sum_0^\infty x^n$.

  For matrices we have $(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots$

  This sum converges if $|\lambda(At)| < 1$.
- I am interested in the function e^{At} and I would like to connect it to S and Λ.
- $e^{At} = I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2} + \frac{S\Lambda^3 S^{-1}t^3}{6} + \cdots + \frac{S\Lambda^n S^{-1}t^n}{n!} + \cdots = Se^{\Lambda t}S^{-1}$

  Question:

  What assumption is built-in to this formula, that is not built to the original formula in the first line?

  Answer:

  The assumption is that A must be diagonalizable.

Diagonal matrix exponentials $e^{\Lambda t}$

- The exponential $e^{\Lambda t}$ of a diagonal matrix is given by:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

- As we already showed

$$\lim_{t\to\infty} e^{\Lambda t} = 0 \text{ if } \mathrm{Re}(\lambda_i) < 0, \forall i$$

$$\lim_{t\to\infty} \Lambda^{t} = 0 \text{ if } |\lambda_i| < 0, \forall i$$

Second order differential equations

- How do I change the second order differential equation

$$y''(t) + by'(t) + ky(t) = 0$$

  to two first order ones?

- I define $u(t) = \begin{bmatrix} y'(t) \\ y(t) \end{bmatrix}$ and therefore,

$$u'(t) = \begin{bmatrix} y''(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y'(t) \\ y(t) \end{bmatrix}$$

$$u'(t) = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} u(t)$$

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Higher order differential equations

- How do I change the n^{th} order differential equation
$$y^{(n)}(t) + b_1 y^{(n-1)}(t) + \cdots + b_{n-1} y(t) = 0$$
to n first order ones?

- I define $u(t) = \begin{bmatrix} y^{(n-1)}(t) \\ \vdots \\ y'(t) \\ y(t) \end{bmatrix}$ and therefore,

$$u'(t) = \begin{bmatrix} y^{(n)}(t) \\ y^{(n-1)}(t) \\ \vdots \\ y''(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -b_1 & -b_2 & \ldots & -b_{n-2} & -b_{n-1} \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} u(t)$$

- As previously analysed the solution of this system is explicitly affiliated to the eigenvalues of the system's matrix.