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Maths for Signals and Systems
Linear Algebra in Engineering

Symmetric Matrices

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Symmetric matrices

- In this lecture we will be interested in symmetric matrices.
- In case of real matrices, symmetry is defined as $A = A^T$.
- In case of complex matrices, symmetry is defined as $A^* = A^T$ or ${A^*}^T = A$. A matrix which possesses this property is called **Hermitian**.
- We can also use the symbol $A^H = {A^*}^T$.
- We will prove that the eigenvalues of a symmetric matrix are real.
- The eigenvectors of a symmetric matrix can be chosen to be orthogonal. If we also choose them to have a magnitude of 1, then the eigenvectors can be chosen to form an orthonormal set of vectors.
- However, the eigenvectors of a symmetric matrix that correspond to different eigenvalues **are** orthogonal (prove is given in subsequent slide).
- For a random matrix with independent eigenvectors we have $A = S\Lambda S^{-1}$.
- For a symmetric matrix with orthonormal eigenvectors we have

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$$

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Real symmetric matrices

Problem:

Prove that the eigenvalues of a symmetric matrix occur in complex conjugate pairs.

Solution:

Consider $Ax = \lambda x$.

If we take complex conjugate in both sides we get

$$(Ax)^* = (\lambda x)^* \Rightarrow A^*\, x^* = \lambda^*\, x^*$$

If A is real then $Ax^* = \lambda^*\, x^*$. Therefore, if λ is an eigenvalue of A with corresponding eigenvector x then λ^* is an eigenvalue of A with corresponding eigenvector x^*.

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Real symmetric matrices cont.

Problem:

Prove that the eigenvalues of a symmetric matrix are real.

Solution:

We proved that if A is real then $Ax^* = \lambda^* x^*$.

If we take transpose in both sides we get

$$x^{*T} A^T = \lambda^* x^{*T} \Rightarrow x^{*T} A = \lambda^* x^{*T}$$

We now multiply both sides from the right with x and we get

$$x^{*T} A x = \lambda^* x^{*T} x$$

We take now $Ax = \lambda x$. We now multiply both sides from the left with x^{*T} and we get

$$x^{*T} A x = \lambda x^{*T} x.$$

From the above we see that $\lambda x^{*T} x = \lambda^* x^{*T} x$ and since $x^{*T} x \neq 0$, we see that $\lambda = \lambda^*$.

Real symmetric matrices cont.

Problem:

Prove that the eigenvectors of a symmetric matrix **which correspond to different eigenvalues** are always perpendicular.

Solution:

Suppose that $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ with $\lambda_1 \neq \lambda_2$.

$$(\lambda_1 x)^T y = x^T \lambda_1 y = (Ax)^T y = x^T A y = x^T \lambda_2 y$$

The conditions $x^T \lambda_1 y = x^T \lambda_2 y$ and $\lambda_1 \neq \lambda_2$ give $x^T y = 0$.

The eigenvectors x and y are perpendicular.

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Complex matrices. Complex symmetric matrices.

- Let us find which complex matrices have real eigenvalues and orthogonal eigenvectors.
- Consider $Ax = \lambda x$ with A possibly complex.
- If we take complex conjugate in both sides we get
$$(Ax)^* = (\lambda x)^* \Rightarrow A^* \, x^* = \lambda^* \, x^*$$
- If we take transpose in both sides we get
$${x^*}^T {A^*}^T = \lambda^* {x^*}^T$$
- We now multiply both sides from the right with x we get
$${x^*}^T {A^*}^T x = \lambda^* {x^*}^T x$$
- We take now $Ax = \lambda x$. We now multiply both sides from the left with ${x^*}^T$ and we get
$${x^*}^T A x = \lambda {x^*}^T x.$$
- From the above we see that if ${A^*}^T = A$ then $\lambda {x^*}^T x = \lambda^* {x^*}^T x$ and since ${x^*}^T x \neq 0$, we see that $\lambda = \lambda^*$.
- If ${A^*}^T = A$ the matrix is called **Hermitian**.

Complex vectors and matrices

- Consider a complex column vector $z = [z_1 \quad z_2 \quad \cdots \quad z_n]^T$.
- Its length is ${z^*}^T z = \sum_{i=1}^{n} |z_i|^2$.
- As already mentioned, when we both transpose and conjugate we can use the symbol $z^H = {z^*}^T$ (Hermitian).
- The inner product of two complex vectors is ${y^*}^T x = y^H x$.
- For complex matrices the symmetry is defined as ${A^*}^T = A$. As already mentioned, these are called Hermitian matrices.
- They have real eigenvalues and perpendicular eigenvectors. If these are complex we check their length using ${q_i^*}^T q_i$ and also ${Q^*}^T Q = I$.

  Example: Consider the matrix

$$A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

  Eigenvalues are found from:

$$(2-\lambda)(5-\lambda) - (3+i)(3-i) = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 - (9 - 3i + 3i - i^2) = 0 \Rightarrow \lambda(\lambda - 7) = 0$$

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Eigenvalue sign

- We proved that:
  - The eigenvalues of a symmetric matrix, either real or complex, are real.
  - The eigenvectors of a symmetric matrix can be chosen to be orthogonal.
  - The eigenvectors of a symmetric matrix that correspond to different eigenvalues are orthogonal.
- **Do not forget the definition of symmetry for complex matrices.**
- It can be proven that the signs of the pivots are the same as the signs of the eigenvalues.
- Just to remind you:

  Product of pivots=Product of eigenvalues=Determinant