Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

Lecture 9, Friday 1st November 2014

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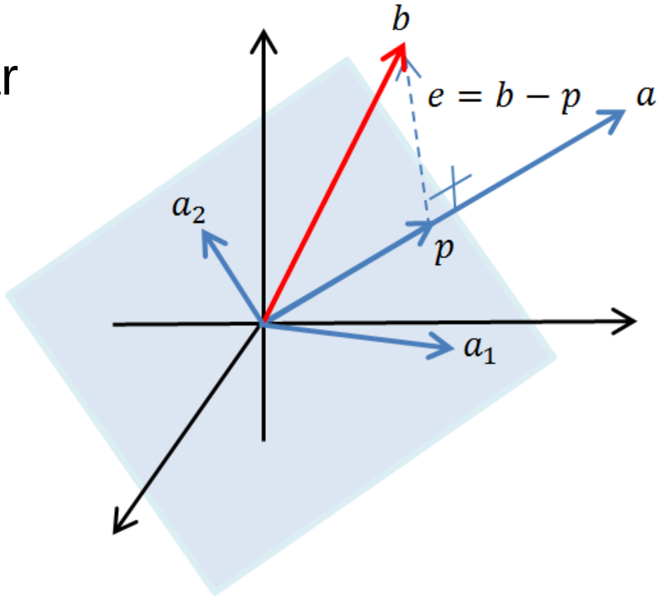
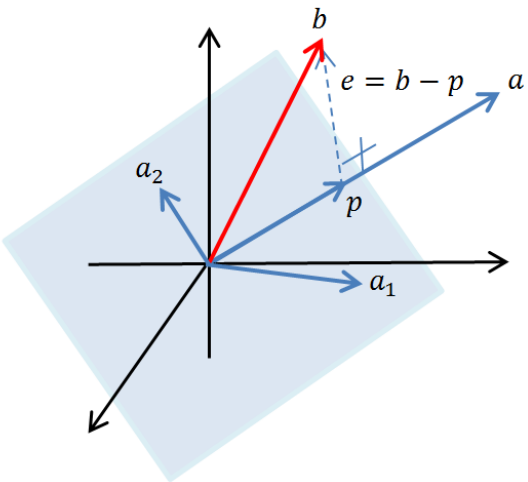
Problem formulation

- Consider again the scenario where the system $Ax = b$ doesn't have solution.
- Goal: Solve $A\hat{x} = p$ instead, where p is the projection of b onto the column space of A.
- The error $e = b - p$ is again perpendicular to the column space of A.
- This scenario is depicted in the figure on the right for R^3.
- The quantities shown are column vectors and $A = [a_1 \quad a_2]$.



Projection error, solution, projection matrix



- The error e is perpendicular to a_1 and a_2.

  $a_1{}^T(b - A\hat{x}) = 0$ and $a_2{}^T(b - A\hat{x}) = 0$

  $$\begin{bmatrix} a_1{}^T \\ a_2{}^T \end{bmatrix}(b - A\hat{x}) = 0 \Rightarrow A^T(b - A\hat{x}) = 0$$
- Consider $A^T(b - A\hat{x}) = 0$.
- **Question:** In which subspace does $(b - A\hat{x})$ belong?

  Answer: $e \in N(A^T)$. Therefore e is perpendicular to the column space of A. This is also obvious from $A^T(b - A\hat{x}) = 0$.
- Solution: $A^T A\hat{x} = A^T\, b \Longrightarrow \hat{x} = (A^T A)^{-1} A^T b$
- Projection: $p = Pb = A(A^T A)^{-1}\, A^T b$
- Projection matrix: $P = A(A^T A)^{-1} A^T$

Projection matrix

- Projection matrix: $P = A(A^T A)^{-1} A^T$

- A is not square (it is rectangular) and therefore we cannot use the property $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$.

- **Question:** If A **<u>was</u>** a square and invertible matrix of size $n \times n$ what would P be?

  Answer: In that case the column space of A would be the entire R^n and therefore the projection of any vector on $C(A)$ would be the vector itself.

  This can be also verified by
  $P = A(A^T A)^{-1} A^T = AA^{-1}(A^T)^{-1} A^T = I \cdot I = I$

Projection matrix

- Projection matrix: $P = A(A^T A)^{-1} A^T$

- **Question:** If b is perpendicular to $C(A)$, what do I get when I apply projection?

  Answer: In that case b belongs to the null space of A^T and therefore $A^T b = 0$. We then obtain $Pb = A(A^T A)^{-1} A^T b = 0$

- **Question:** What is the projection that gives me e?

  Answer: $P'b = e \Rightarrow P'b = b - p \Rightarrow P'b = b - Pb$

  $\Rightarrow P'b = Ib - Pb = (I - P)b \Rightarrow P' = I - P$

Projection matrix: Properties

- Properties of P

➢Symmetric

$A^T A$ symmetric and therefore $(A^T A)^{-1}$ is symmetric

[to prove this we use the property $(A^{-1})^T = (A^T)^{-1}$]

$P^T = [A(A^T A)^{-1} A^T]^T = (A^T)^T [(A^T A)^{-1}]^T A^T = A\ (A^T A)^{-1} A^T = P$

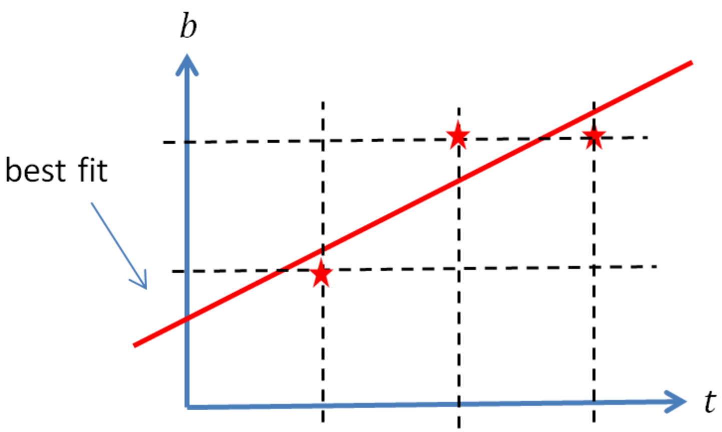
➢$P^2 = P$

$P^2 = A\ (A^T A)^{-1} A^T A\ (A^T A)^{-1} A^T = A\ (A^T A)^{-1} [A^T A\ (A^T A)^{-1}] A^T =$

$= A\ (A^T A)^{-1} I A^T = A\ (A^T A)^{-1} A^T = P$

Application: Least squares method. Fitting by a line.

- Problem: I am given the three points shown with stars in the figure on the right. I want to fit them on the "best" possible straight line.
- The points are (1,1), (2,2), (3,2)
- The required line is $b = C + Dt$ with C and D unknowns.



- The three points must satisfy the line equation:

$$C + D = 1$$
$$C + 2D = 2$$
$$C + 3D = 2$$

Application: Least squares method. Fitting by a line.

- Problem: Solve the system $Ax = b$

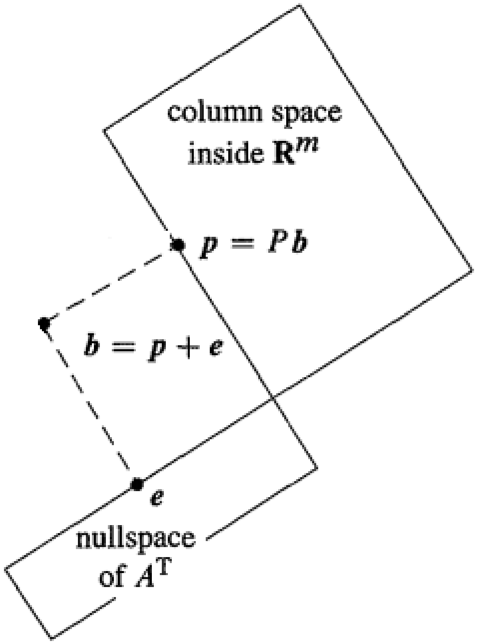
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

- The system is not solvable because $b \notin C(A)$ (show that).
- Solution: Solve $A\hat{x} = p$ instead, where p is the projection of b onto $C(A)$.

  A random b is written as:

  $b - p = e \Rightarrow b = p + e$

  p is in column space and e is perpendicular to the column space.
- Projection kills e and keeps p.



Solution of the specific example

- The proposed approach $A\hat{x} = p$ yields the solution which can be also obtained if we look for an $\hat{x}$ that minimizes the function
$$\|A\hat{x} - b\|^2 = \|e\|^2$$
- The above function is the magnitude of the total error.
- This method is called **linear regression**.
- Let's solve this specific problem at the end!

$A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$ $A^T b = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

- We can use the inverse $(A^T A)^{-1} = \begin{bmatrix} 7/3 & -1 \\ -1 & 1/2 \end{bmatrix}$
- Or we can solve directly the equations $3C + 6D = 5, 6C + 14D = 11$.
- Final solution is $D = \frac{1}{2}, C = \frac{2}{3}$. The "best" line is $b = \frac{2}{3} + \frac{1}{2}t$ (red line).

Solution of the specific example

- As mentioned, an alternative approach is to find the unknowns that minimize the error function:

$$(C+D-1)^2+(C+2D-2)^2+(C+3D-2)^2$$

- We must take the partial derivatives with respect to the two unknowns and set them to zero.
- Verify that, by implementing this method, you get the same solution as previously.
- The vector p is obtained by:

$$p_1 = C + D = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$p_2 = C + 2D = \frac{2}{3} + 1 = \frac{5}{3}$$

$$p_3 = C + 3D = \frac{2}{3} + \frac{3}{2} = \frac{13}{6}$$

Solution of the specific example

- Furthermore:

$$e_1 = \frac{-1}{6}$$
$$e_2 = \frac{2}{6}$$
$$e_3 = \frac{-1}{6}$$

- Verify that:
  - $b = p + e$
  - p and e are perpendicular to each other

Problem

- **Problem:** Show that if A has independent columns then $A^T A$ is invertible. I will need this in the next lecture.
- I must prove that $A^T A x = 0$ implies $x = 0$. I assume then that $A^T A x = 0$.
- The above implies $x^T A^T A x = 0 \Rightarrow (Ax)^T A x = 0$
- I define $Ax = y$ and therefore $y^T y = 0 \Rightarrow ||y||^2 = 0 \Rightarrow y = 0 \Rightarrow Ax = 0 \Rightarrow x = 0$ if A has independent columns!
- There is one case in which the columns of A are for sure independent and this is when they are perpendicular unit vectors!