Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

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Determinants

- The **Determinant** is a crucial number associated with square matrices only.

- It is denoted by $\det(A) = |A|$. These are two different symbols we use for determinants.

- If a matrix A is invertible, that means $\det(A) \neq 0$.

- Furthermore, $\det(A) \neq 0$ means that matrix A is invertible.

- For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant is defined as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. This formula is explicitly associated with the solution of the system $Ax = b$ where A is a 2×2 matrix.

Properties of determinants cont.

1. $\det(I) = 1$. This is easy to show in the case of a 2×2 matrix using the formula of the previous slide.

2. If we exchange two rows of a matrix the sign of the determinant reverses. Therefore:
   - If we perform an even number of row exchanges the determinant remains the same.
   - If we perform an odd number of row exchanges the determinant changes sign.
   - Hence, the determinant of a **permutation** matrix is 1 or -1.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{ and } \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \text{ as expected.}$$

Properties of determinants cont.

3a. If a row is multiplied with a scalar, the determinant is multiplied with that scalar too, i.e., $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

3b. $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Note that $\det(A + B) \neq \det(A) + \det(B)$

I observe "linearity" only for a single row.

4. Two equal rows leads to $\det = 0$.
   - As mentioned, if I exchange rows the sign of the determinant changes.
   - In that case the matrix is the same and therefore, the determinant should remain the same.
   - Therefore, the determinant must be zero.
   - This is also expected from the fact that the matrix is not invertible.

Properties of determinants cont.

5. $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
   Therefore, the determinant after elimination remains the same.

6. A row of zeros leads to $\det = 0$. This can verified as follows for any matrix:
$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 \cdot a & 0 \cdot b \\ c & d \end{vmatrix} = 0\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

7. Consider an upper triangular matrix ($*$ is a random element)
$$\begin{vmatrix} d_1 & * & \dots & * \\ 0 & d_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{vmatrix} = d_1 d_2 \dots d_n$$
   I can easily show the above using the following steps:
   - I transform the upper triangular matrix to a diagonal one using elimination.
   - I use property 3a n times.
   - I end up with the determinant $\prod_{i=1}^{n} d_i \det(I) = \prod_{i=1}^{n} d_i$.
   - The same observations are valid for a lower triangular matrix.

Properties of determinants cont.

8. $\det(A) = 0$ when A is singular. This is because if A is singular I get a row of zeros by elimination.

   Using the same concept I can say that if A is invertible then $\det(A) \neq 0$.

   In general I have $A \to U \to D$, $\det(A) = d_1 d_2 \dots d_n$ =product of pivots.

9. $\det(AB) = \det(A)\det(B)$

   $\det(A^{-1}) = \dfrac{1}{\det(A)}$

   $\det(A^2) = [\det(A)]^2$

   $\det(cA) = c^n \det(A)$ where $A: n \times n$ and c is a scalar.

10. $\det(A^T) = \det(A)$.
    - We use the LU decomposition of A, $A = LU$. Therefore, $A^T = U^T L^T$.
    - L and L^T have the same determinant due to the fact that they are triangular matrices (so it is the product of the diagonal elements). The same observation is valid for U and U^T. This observation and the property $\det(AB) = \det(A)\det(B)$ yields $\det(A^T) = \det(A)$.
    - This property can also be proved by the use of **induction**.

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Determinant of a 2×2 matrix

- The goal is to find the determinant of a 2×2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ using the properties described previously.
- We know that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$.
- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} =$
  $0 + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + 0 = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc$
- I can realize the above analysis for 3×3 matrices.
- I break the determinant of a 2×2 random matrix into 4 determinants of simpler matrices.
- In the case of a 3×3 matrix I break it into 27 determinants.
- And so on.

Determinant of any matrix

- For the case of a 2×2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ we got:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0 + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + 0$$

- The determinants which survive have strictly one entry from each row and each column.
- The above is a universal conclusion.

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Determinant of any matrix cont.

- For the case of a 3 × 3 matrix $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ we got:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \cdots =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}\, a_{32} + \cdots$$

- As mentioned the determinants which survive have strictly one entry from each row and each column.
- Each of these determinants is obtained by the product its non-zero elements with a plus or a minus sign in front. The sign is determined by the number of re-orderings required so that the matrix of interest becomes diagonal. For example the second determinant shown above requires one re-ordering (swap of rows 2 and 3) to become the determinant of a diagonal matrix. One re-ordering implies, as shown previously, negating the sign of the original determinant.

Determinant of any matrix cont.

- For the case of a 2×2 matrix the determinant has 2 survived terms.
- For the case of a 3×3 matrix the determinant has 6 survived terms.
- For the case of a 4×4 matrix the determinant has 24 survived terms.
- For the case of a $n \times n$ matrix the determinant has $n!$ survived terms.
  - ➢ The elements from the first row can be chosen in n different ways.
  - ➢ The elements from the second row can be chosen in $(n-1)$ different ways.
  - ➢ and so on…

Problem

Find the determinant of the following matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(The answer is 0).

The general form for the determinant

- For the case of a $n \times n$ matrix the determinant has $n!$ terms.

$$\det(A) = \sum_{n!\text{terms}} \pm a_{1a} a_{2b} a_{3c} \dots a_{nz}$$

  - $a, b, c, \dots, z$ are different columns.
  - In the above summation, half of the terms have a plus and half of them have a minus sign.

The general form for the determinant cont.

- For the case of a $n \times n$ matrix, the method of **cofactors** is a technique which helps us to connect a determinant to determinants of smaller matrices.

$$\det(A) = \sum_{n!\text{terms}} \pm a_{1a}a_{2b}a_{3c} \dots a_{nz}$$

- For a 3×3 matrix we have $\det(A) = a_{11}(a_{22}a_{33} - a_{23}\, a_{32}) + \cdots$
- $a_{22}a_{33} - a_{23}\, a_{32}$ is the determinant of a 2×2 matrix which is a sub-matrix of the original matrix.

Cofactors

- The **cofactor** of element a_{ij} is defined as follows:

$$C_{ij} = \pm\det\left[(n-1)\times(n-1) \text{ matrix } A_{ij}\right]$$

  - A_{ij} is the $(n-1)\times(n-1)$ that is obtained from the original matrix if row i and column j are eliminated.
  - We keep the $+$ if $(i+j)$ is even.
  - We keep the $-$ if $(i+j)$ is odd.

- Cofactor formula along row 1:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}\, C_{1n}$$

- Generalization:
  - Cofactor formula along row i: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}\, C_{in}$
  - Cofactor formula along column j: $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}\, C_{nj}$

- **Cofactor formula along any row or column can be used for the final estimation of the determinant.**

Estimation of the inverse A^{-1} using cofactors

- For a 2×2 matrix it is quite easy to show that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- The general formula for the inverse A^{-1} is given by:

$$A^{-1} = \frac{1}{\det(A)} C^T$$
$$AC^T = \det(A) \cdot I$$

- C_{ij} is the cofactor of a_{ij} which is a sum of products of $(n-1)$ entries.
- In general

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \det(A) \cdot I$$

Solve $Ax = b$ when A is square and invertible

- The solution of the system $Ax = b$ when A is square and invertible can be now obtained from

$$x = A^{-1}b = \frac{1}{\det(A)} C^T b$$

$$C^T b = \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

- Lets find the first element of vector x. This is:

$$x_1 = \frac{1}{\det(A)} (b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1})$$

- We see that in $(b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1})$ the cofactors used are the same ones that we use when we calculate the determinant of A using its first column.
- Therefore, $(b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}) = \det(B_1)$ with

$$B_1 = [b \, \vdots \quad \text{last } (n-1) \text{ columns of } A]$$

- B_1 is obtained by A if we replace the first column with b.
- In general, B_i is obtained by A if we replace the ith column with b.

Solve $Ax = b$ when A is square and invertible cont.

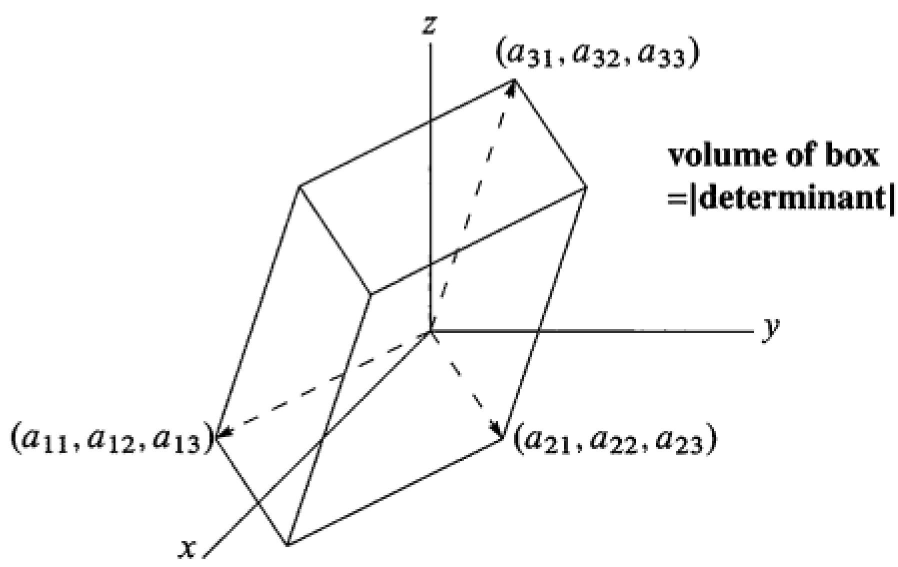
- **Cramer's rule:**
  - The solution of the system $Ax = b$ with $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is given by

$$x_i = \frac{\det(B_i)}{\det(A)}, \; i = 1, \dots, n.$$

  - B_i is obtained by A if we replace the ith column with b.
  - In practice we must find $(n+1)$ determinants.

Geometrical interpretation of the determinant

- Consider A to be a matrix of size 2×2.
- It can be proven that the determinant of A is the area of the parallelogram with the column vectors of A as the two of its sides.
- Consider A to be a matrix of size 3×3.
- It can be proven that the determinant of A is the volume of the parallelepiped with the column vectors of A as the three of its sides.
- This observation is extended to matrices of dimension $n \times n$ where the determinants are **parallelotopes**.



Geometrical interpretation of the determinant cont.

- Take $A = I$. Then the parallelepiped mentioned previously is the unit cube and its volume is 1.

Problem:

Consider an orthogonal square matrix Q. Prove that $\det(Q) = 1$ or -1.

Solution:

$Q^T Q = I \Rightarrow \det(Q^T Q) = \det(Q^T)\det(Q) = \det(I) = 1$

But $\det(Q^T) = \det(Q) \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$

- Take Q and double one of its vectors.
  - The determinant doubles as well (property 3a).
  - In that case, the cube's volume doubles, i.e., you have two cubes sitting on top of each other.

You may download for free a demonstration by Wolfram that illustrates the above observations http://demonstrations.wolfram.com/DeterminantsSeenGeometrically/.