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Maths for Signals and Systems

Linear Algebra in Engineering

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In this set of lectures we will talk about:

- Spaces other than vector spaces.
- Orthogonal subspaces.
- Projections on to spaces.
- How to solve the problem $Ax = b$ when b does not belong in the column space of A.
- Lease Squares Minimization or Linear Regression.

Generalization of the concept of space

- Consider a set of entities (objects) that are not necessarily $1-D$ vectors.
- Assume that multiplication with a scalar and addition can be defined for these entities.
- A new space can be defined from all linear combinations of such entities.

Example: Space of matrices

- Consider a space M which contains all 3×3 matrices.
- Addition and multiplication with a scalar can be applied to matrices, e.g., if A and B are matrices, then $A+B$ and cA are matrices too.
- Examples of subspaces of M:
  - All upper triangular matrices.
  - All lower triangular matrices.
  - All symmetric matrices.
  - All diagonal matrices.

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Mathematics for Signals and Systems-Space of Matrices

- Consider the space M of all 3×3 matrices.
  - ➢ The dimension of this space is 9, i.e., $\dim(M) = 9$.
  - ➢ The simplest basis of this space is:

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\dots\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

  - ➢ Any 3×3 matrix can be written as a linear combination of the above matrices.

- Consider the subspace S of all 3×3 **symmetric** matrices. **(Keep in mind that symmetry implies square matrices only)**.
  - ➢ The dimension of this subspace is 6, i.e., $\dim(S) = 6$.
  - ➢ The simplest basis of this subspace is:

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

  - ➢ Any 3×3 symmetric matrix can be written as a linear combination of the above matrices.

Space of Matrices cont.

- Consider the subspace U of all 3×3 upper triangular matrices.
  - The dimension of this subspace is 6, i.e., $\dim(U) = 6$.
  - The simplest basis of this subspace is:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

- Consider the subspace of the intersection of symmetric and upper triangular matrices $S \cap U$. **This subspace consists of diagonal matrices.**
  - The dimension of this subspace is 3, i.e., $\dim(S \cap U) = 3$.
  - A basis is:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

- **Note that the union $S \cup U$ is not a subspace.**

Space of Matrices cont.

- As mentioned the union $S \cup U$ is not a subspace.

- Lets consider $S + U$.

  Question:

  What matrices can I get from $S + U$ where both S and U are of dimension $n \times n$?

  Answer:

  I can basically get any $n \times n$ matrix. **Can you possibly prove why?**

- We notice that for $n = 3$ we have:

$$\dim(S + U) = 9$$
$$\dim(S) = 6$$
$$\dim(U) = 6$$

- In general we can prove that:

$$\dim(S) + \dim(U) = \dim(S + U) + \dim(S \cap U)$$

Rank One Matrices

- An example of a 2×3 matrix of rank equal to 1 is:

$$A=\begin{bmatrix}1 & 4 & 5\\ 2 & 8 & 10\end{bmatrix}$$

- A basis of the row space of the above matrix is the vector $\begin{bmatrix}1 & 4 & 5\end{bmatrix}$.

- A basis of the column space of the above matrix is the vector $\begin{bmatrix}1\\ 2\end{bmatrix}$.

- Note that the well known property holds $\dim C(A)=r=\dim C(A^T)$.

- In this example $r=1$.

Rank One Matrices cont.

- We can write the previous matrix A as:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}.$$

- Every rank one matrix can be decomposed into the product of a column vector and a row vector $A = v\,w^T$, with v and w column vectors.

- Furthermore, if v and w are column vectors, the matrix $v\,w^T$ is always of rank 1.

- All matrices can be written as a combination of rank one matrices.

- Rank one matrices are the building blocks for all matrices!

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Orthogonal subspaces: Definition and a couple of questions

- Suppose that a subspace S is orthogonal to a subspace T. **What does this mean?** It actually means that every vector in S is orthogonal to every vector in T.

- Assume we are in R^3. Consider two 2-D planes which are perpendicular to each other (for example a wall and the floor). Are these planes orthogonal?
  (The answer is NO: the line which is their intersection belongs to both)

- Assume we are in R^2.
  1. When is a line through the origin orthogonal to the entire plane?
  2. When is a line through the origin orthogonal to the 0 subspace?
  3. When is a line through the origin orthogonal to another line through the origin?

  (answers: 1. never, 2. always, 3. when they form a 90^o angle)

Orthogonal subspaces: more questions

- The row and null space of a matrix A of size $m \times n$ are orthogonal. **Why?**
  Vectors which satisfy $Ax = \mathbf{0}$ are orthogonal to all rows of A.

- Assume we are in R^3: consider two orthogonal lines. Could their subspaces be the row space and the null space of a matrix?
  (The answer is NO: dim$C(A^T)$+dim$N(A) = r + (3 - r) = 3$)

- Row space and null space are orthogonal and furthermore:
  - Their dimensions add up to the size of rows (or number of columns). This is basically the size of the maximum space that the rows can form.
  - Row space and null space are orthogonal complements in R^n.

Solve a system when there is no exact solution

- "Solve" $Ax = b$ when there is no exact solution, i.e., $b \notin C(A)$.

- We will realize, in subsequent sections, that there is a matrix which will play an important role for the solution to this problem: This is $A^T A$. It has the following properties:
  - It is square of size $n \times n$ (A is of size $m \times n$).
  - It is symmetric because $(A^T A)^T = A^T (A^T)^T = A^T A$.
  - It is invertible if A has n independent columns, i.e., the null space of A is zero (proof follows later).

Example 1: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$. Show that $A^T A$ is invertible.

Example 2: $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$. Show that $A^T A$ is NOT invertible.

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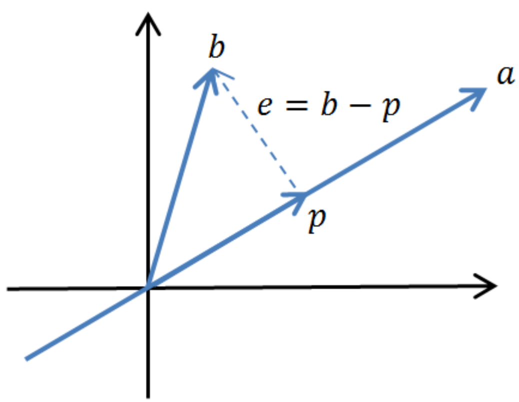
Projection matrix

PROBLEM: Find the projection of a vector b onto a line a. We need this to solve the problem $Ax = b$, $b \notin C(A)$. **(All vectors are assumed column vectors.)**

- This is a point p on the straight line formed by vector a, such that $p = xa$, where x is a scalar.
- p is defined such as the vector $e = b - p$ is orthogonal to the line formed by vector a.
- Inner product $a^T e = a^T(b - p) = a^T(b - xa) = 0$ or $x = \frac{a^T b}{a^T a}$.
- Based on the above $p = \frac{a^T b}{a^T a} a = \frac{aa^T}{a^T a} b = Pb$ with $P = \frac{aa^T}{a^T a}$ **(observe the form of P)**.
- Note that $a^T a$ is a scalar and aa^T is a square matrix.
- P is called the **projection matrix**.



Projection matrix properties

- **What is the column space of P?**

  It is the subspace formed by vector a. The reason is that if $a = [a_1 \quad a_2]^T$ then $P = \frac{1}{\|a\|^2}\begin{bmatrix} a_1{}^2 & a_1a_2 \\ a_1a_2 & a_2{}^2 \end{bmatrix}$ and therefore, $C(P)$: $c_1\begin{bmatrix} a_1{}^2 \\ a_1a_2 \end{bmatrix} + c_2\begin{bmatrix} a_1a_2 \\ a_2{}^2 \end{bmatrix} = (c_1a_1 + c_2a_2)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = c\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

- **What is the rank of P?**

  Obviously 1.

- **What happens if I do the projection twice?**

  Nothing should happen. Therefore, the condition $P^2 = P$ must hold. Verify this using the above 2×2 matrix.

- **If we double b what happens to the projection?**

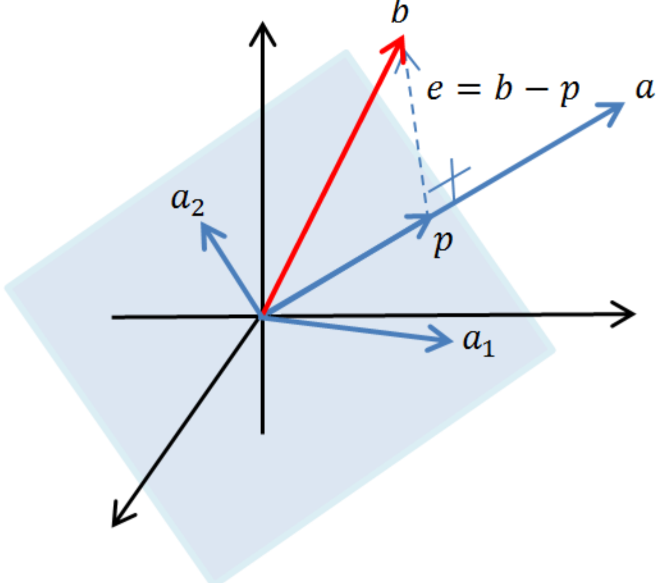
  The projection is doubled too.

- **If we double a what happens to the projection?**

  Nothing should happen to the projection in that case.

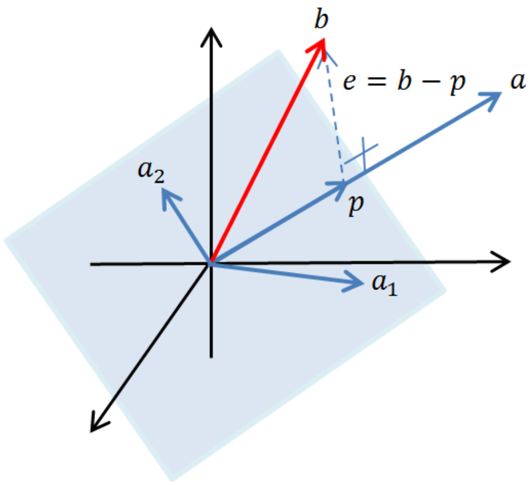
Solving Ax=b using projections

- Consider again the scenario where the system $Ax = b$ doesn't have solution.
- Goal: Solve $A\hat{x} = p$ instead, where p is the projection of b onto the column space of A.
- As already mentioned, the projection p of b onto the column space of A is found by forcing the error $e = b - p$ to be perpendicular to the column space of A.
- This scenario is depicted in the figure below for R^3.
- The quantities shown are column vectors and $A = [a_1 \quad a_2]$.



Projection error, solution and projection matrix

- The error e is perpendicular to a_1 and a_2.



$a_1{}^T(b - A\hat{x}) = 0$ and $a_2{}^T(b - A\hat{x}) = 0$

$$\begin{bmatrix} a_1{}^T \\ a_2{}^T \end{bmatrix}(b - A\hat{x}) = \mathbf{0} \Rightarrow A^T(b - A\hat{x}) = \mathbf{0}$$

Question:

Consider $A^T(b - A\hat{x}) = \mathbf{0}$.

In which subspace does $e = (b - A\hat{x})$ belong?

Answer:

Since $A^T(b - A\hat{x}) = \mathbf{0}$, we observe that $e \in N(A^T)$ (left nullspace). Therefore e is perpendicular to the column space of A.

- Solution of the "new" system: $A^T A\hat{x} = A^T b$
- If $A^T A$ is invertible then: $\hat{x} = (A^T A)^{-1} A^T b$
- Projection: $p = A\hat{x} = A(A^T A)^{-1} A^T b = Pb$
- Projection matrix: $P = A(A^T A)^{-1} A^T$

Projection matrix

- Projection matrix: $P = A(A^T A)^{-1} A^T$

- In general, A is not square (it is rectangular) and therefore, we cannot use the property $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$.

Question:

If A **was** a square and invertible matrix of size $n \times n$ what would P be?

Answer:

In that case the column space of A would be the entire R^n and therefore the vector b would belong to $C(A)$. Therefore, the projection of any vector b on $C(A)$ would be the vector itself. This can be also verified by:

$$P = A(A^T A)^{-1} A^T = AA^{-1}(A^T)^{-1} A^T = I \cdot I = I$$

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Projection matrix: Properties

- Properties of P.

  - Symmetric
    $A^T A$ symmetric and therefore $(A^T A)^{-1}$ is symmetric
    [to prove this we use the property $(A^{-1})^T = (A^T)^{-1}$]
    $P^T = [A(A^T A)^{-1} A^T]^T = (A^T)^T [(A^T A)^{-1}]^T A^T = A\,(A^T A)^{-1} A^T = P$

  - $P^2 = P$
    $P^2 = A\,(A^T A)^{-1} A^T A\,(A^T A)^{-1} A^T = A\,(A^T A)^{-1} [A^T A\,(A^T A)^{-1}] A^T =$
    $= A\,(A^T A)^{-1} I A^T = A\,(A^T A)^{-1} A^T = P$

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Least Squares Minimization

Problem:

Show that the proposed approach $A\hat{x} = p$ yields the solution which can be also obtained if we look for an $\hat{x}$ that minimizes the function:

$$\|A\hat{x} - b\|^2 = \|e\|^2$$

Solution:

We are looking to minimize the function $\|A\hat{x} - b\|^2 = \|e\|^2 = (A\hat{x} - b)^T(A\hat{x} - b)$. This is a function of $\hat{x}$ only and therefore, we can formulate the problem:

$$\min f(\hat{x}) = \min[(A\hat{x} - b)^T(A\hat{x} - b)]$$

A function is minimized at points for which its first derivative is zero. Therefore, we are looking for the $\hat{x}$s that satisfy the equation:

$$\frac{\partial f(\hat{x})}{\partial \hat{x}} = \mathbf{0} \Rightarrow \frac{\partial}{\partial \hat{x}}(A\hat{x} - b)^T(A\hat{x} - b) = \mathbf{0}$$

$$(A\hat{x} - b)^T(A\hat{x} - b) = [(A\hat{x})^T - b^T](A\hat{x} - b) = (\hat{x}^T A^T - b^T)(A\hat{x} - b)$$

$$= \hat{x}^T A^T A\hat{x} - \hat{x}^T A^T b - b^T\ A\hat{x} + b^T b$$

$$\frac{\partial}{\partial \hat{x}}(\hat{x}^T A^T A\hat{x} - \hat{x}^T A^T b - b^T\ A\hat{x} + b^T b) = 2A^T A\hat{x} - A^T b - A^T\ b + \mathbf{0} = 2A^T A\hat{x} - 2A^T b$$

Therefore, the derivative is zero if $A^T A\hat{x} = A^T b$. This is equal to the previous solution. **This approach is called Least Squares Minimization.**