Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

Lectures 4-5, Tuesday 18th October 2016

DR TANIA STATHAKI

READER (ASSOCIATE PROFFESOR) IN SIGNAL PROCESSING

IMPERIAL COLLEGE LONDON

Mathematics for Signals and Systems

In this set of lectures we will talk about:

- The **complete** or **general** solution of the system $Ax = b$.
- Independent vectors.
- Dependent vectors.
- Basis of a space.
- Dimension of a space.

Mathematics for Signals and Systems

- In the previous set of lectures, we discussed about the system $Ax = 0$.
- In this lecture, we are interested in the general case, $Ax = b$, where the right-hand side is not zero.
- A system of linear equations, $Ax = b$, can have one solution, infinite solutions or no solution.
- Previously we discussed about **pivot** and **free** variables.
- Intuitively, we know that we have free variables if the number of equations is less than the number of unknowns. Furthermore, we know that if the number of equations is less than the number of unknowns the system has infinite number of solutions.
- Intuitively, we know that if the number of equations is larger than the number of unknowns the system, in general, hasn't got exact solutions or it might have one solution.

Solution of the general form Ax=b
General=Particular+Homogeneous

- Let's focus on the system $Ax = b$.
- If x_1 and x_2 are solutions of the system $Ax = b$ then $(x_2 - x_1)$ and $(x_1 - x_2)$ are solutions of the system $Ax = 0$.
- We see that $x_2 = x_1 + (x_2 - x_1)$ and $x_1 = x_2 + (x_1 - x_2)$
- From the above, we observe that if we pick a specific solution x_1 of the system $Ax = b$, any other solution x_2 can be expressed as the previous solution x_1 plus a solution of the system $Ax = 0$.
- Therefore, any solution of the system $Ax = b$ consists of two parts as follows:
  - Any specific solution of the system $Ax = b$. This is called **particular solution**. One way to obtain a particular solution, which we denote with x_p, is by setting all free variables to zero and solving for the pivot variables. Hence, we have that $Ax_p = b$.
  - The (family of) solution(s) of the system $Ax = 0$. We denote these solutions with x_n. Hence, we have that $Ax_n = 0$. The system $Ax = 0$ is called **homogeneous system** and its solutions are called **homogeneous solutions**.

Solution of the general form Ax=b

- In this lecture we will discuss when the system $Ax = b$ has solution(s) and find this(these) solution(s).
- As already mentioned, systems of linear equations can have **one solution**, **infinite solutions** or **no solution**.
- Let us consider the system $Ax = b$, where A is taken from the previous lecture:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

- We will perform **elimination** for the general case where b is a non-zero vector, so the augmented matrix is:

$$\begin{matrix} [2]-2[1] \\ [2]-3[1] \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \quad [3]-[2] \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

- We can immediately spot that the above system of linear equations can be solved only if $b_3 - b_2 - b_1 = 0$.
- The above formula is a **condition for solvability**.

Solution of the general form Ax=b. Particular Solution.

- We have seen in the previous example that the condition for solvability is $b_3 - b_2 - b_1 = 0$.

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

- Let us consider the complete solution of the above system for a vector b that satisfies the above condition, for example $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$.
- In that case the system becomes:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 1$$
$$2x_3 + 4x_4 = 3$$

- The pivot variables of the above system are x_1 and x_3, and the free variables x_2 and x_4.
- By setting all free variables to zero we have:

$$x_1 + 2x_3 = 1$$
$$2x_3 = 3$$

Particular and Homogeneous Solutions cont.

- Based on the previous slide, the pivot variables of the system are obtained by solving the equations below.

$$x_1 + 2x_3 = 1$$
$$2x_3 = 3$$

The pivot variables are $x_3 = 3/2$, $x_1 = -2$.

- Therefore, the particular solution of the system is

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

- Furthermore, as explained previously, the family of solutions of the homogeneous system $Ax = 0$ is given by:

$$x_n = c_1 x_1^s + c_2 x_2^s$$

where $x_1^s,\ x_2^s$ are the special solutions $x_1^s = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2^s = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

Imperial College London

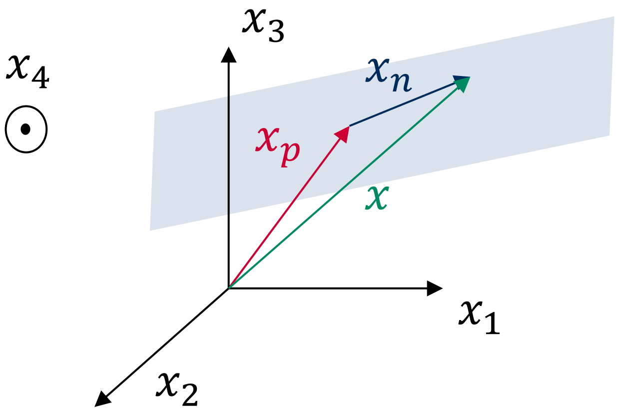
Particular and Homogeneous Solutions cont.

- The complete (general) solution of the system $Ax = b$ is $x = x_p + x_n$.

- For the particular example presented previously the complete solution is

$$x = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

- Therefore, the complete solution is a family of solutions which form a two dimensional plane in R^4 that goes through the point x_p.



Imperial College London

Solution of the general form Ax=b. More rows than columns.

- Consider an $m \times n$ matrix A with $n < m$ *and* rank r.
- We know that $r \leq m$ and $r \leq n$.
- Suppose that the condition of **full column rank**, i.e., $r = n$ holds. In that case:
  - We don't have free variables.
  - The null space $N(A)$ contains only the zero vector.
  - **IF** there is a solution to the system $Ax = b$, it is the particular solution x_p.
  - Therefore, the system has a unique solution, if this exists.
  - We conclude that the number of solutions is 0 or 1.

Example: Consider the system with matrix A shown below.

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}} \xrightarrow{elimin.} \underset{R}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

- rank = 2
- there aren't any free variables
- the null space is the zero vector
- there is a unique solution to $Ax = b$ only when b is a linear combination of the columns of A
- this unique solution is the particular solution x_p

Solution of the general form Ax=b. More columns than rows.

- Consider an $m \times n$ matrix A with $n > m$ *and* rank r.
- We know that $r \leq m$ and $r \leq n$.
- Suppose that the condition of **full row rank**, i.e., $r = m$ holds. In that case:
  - We have $n - m$ free variables.
  - The null space $N(A)$ is non-zero.
  - The number of solutions is ∞.

Example: Consider the system with matrix A shown below.

$$\underset{A}{\begin{bmatrix} 1 & 4 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}} \xrightarrow{elimin.} \underset{R}{\begin{bmatrix} 1 & 0 & f_{12} & f_{12} \\ 0 & 1 & f_{21} & f_{22} \end{bmatrix}}$$

- rank = 2
- there are 2 free variables
- the null space is non-zero
- there are infinite number of solutions

- Note that in the above example I assumed that the pivot columns are gathered together in the left part of the RREF matrix, so that $R = [I \ F]$.
- I can always rearrange the order of my unknowns so that $R = [I \ F]$.

Solution of the general form Ax=b. Same rows and columns.

- Consider an $m \times n$ matrix A with $m = n$ *and* rank r.
- We know $r \leq m$ and $r \leq n$.
- Suppose that the condition of **full row and column rank**, i.e., $r = m = n$ holds.
  - We have 0 free variables.
  - The null space $N(A)$ contains only the zero vector.
  - A solution always exists.
  - The solution is unique.

Example: Consider the system with matrix A showing below.

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}} \xrightarrow{elimin.} \underset{R}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

- rank = 2
- there are 0 free variables
- the null space is the zero vector
- A is invertible and the corresponding echelon matrix R is the identity matrix
- there is always a unique solution

Imperial College London

Linear Independence

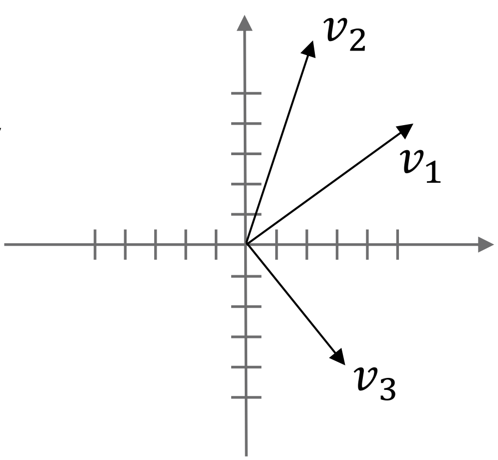
- The vectors $v_1, v_2, v_3, .. v_n$ are **independent** if no linear combination of them gives the zero vector (except the zero combination, all $c_i = 0$)

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + .. c_n v_n \neq 0$$

- Take two non-zero, non-parallel vectors in the two dimensional space.
- They are independent because there isn't any linear combination with non-zero coefficients of them that can give the zero vector.
- Now consider three vectors in the two dimensional space. These are dependent. That means there exist x_i, $i = 1,2,3$ for which

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

- In other words, the above formula consists a system with two equations and three unknowns, so there are definitely free variables and the nullspace is nonzero.



Therefore, there are vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for which $Ax = 0$

(A has the vectors v_i in its columns).

Linear Independence of Column Vectors

- Assume that the vectors $v_1, v_2, v_3, .. \, v_n$ are the columns of a matrix A and they are independent. Assume that the above vectors have m elements. In that case $m \geq n$.
- The rank of A is n.
- The rows of A are vectors of n elements.
- Since the rank of A is n the rows form a basis of the $n -$ dimensional space.
- Therefore, the nullspace $N(A)$ is only the zero vector.
- In other words $Ax = 0$ has no solutions other than the zero vector.
- In this case there are no free variables.

On the other hand:

- The vectors $v_1, v_2, v_3, .. \, v_n$ are dependent if $Ax = 0$ has solutions other than zero, i.e. $N(A)$ is a non-zero subspace.
- If a matrix A has dependent columns then $rank(A) < n.$
- In this case there are free variables.

Span

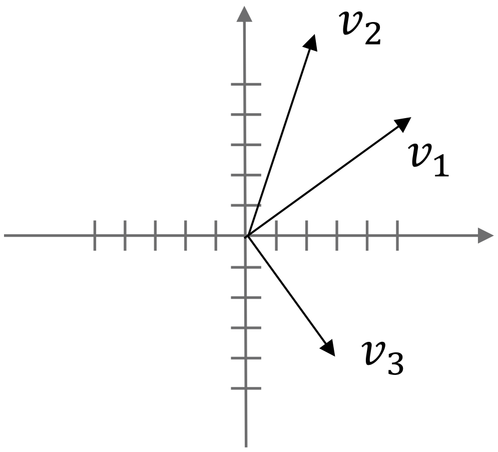
- Vectors $v_1, v_2, v_3, .. v_n$ **span a space**. This means that there is a space that consists of all combinations of those vectors.

- The columns of a matrix span the column space.

- Let S be the space that a set of vectors span. That space is the smallest space with those vectors in it.

Example:

- Vectors v_1, v_2 and v_3 span the R^2.
- R^2 is the smallest space with v_1, v_2 and v_3 in it.



Imperial College London

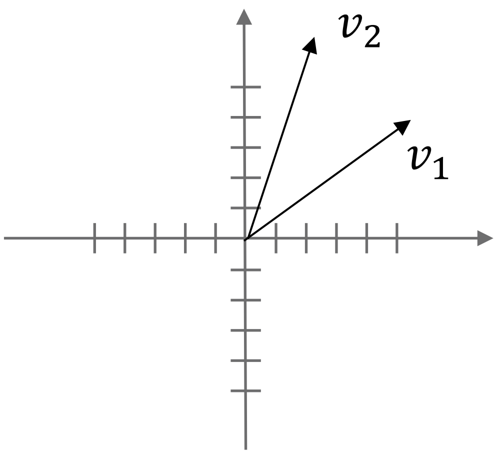
Basis

- A **basis for a vector space** is a sequence of vectors $v_1, v_2, v_3, .. v_n$ with two properties:

1. They are independent.
2. They span the space.

Example:

- Vectors v_1 and v_2 span the R^2.
- The independent vectors v_1 and v_2 form a basis.



Basis cont.

- Consider the hyperplane R^n. n vectors consist a basis if the $n \times n$ matrix with these vectors as columns or rows is invertible.
- Consequently, the columns or rows of an invertible $n \times n$ matrix form a basis of R^n.
- Every basis of R^n has the same number of vectors, which is n.

- The number of vectors that form a basis of a space R^n is the **dimension** of that space.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

- The columns of A span the column space of A.
- In that example they are not independent.
- A basis for the column space consists of the first two columns.
- The dimension of the column space is 2 and therefore, $rank(A) = 2$.
- $N(A)$ has solutions other than the zero vector.

Basis cont.

- We have seen that the rank of the matrix is the dimension of the columns space:

$$dimC(A) = r$$

- The dimension of the nullspace is the same as the number of the free variables.

$$dimN(A) = n - r$$

- The vectors of the special solutions form a basis for the nullspace.

Imperial College London

The Four Fundamental Subspaces: Column Space, Row Space, Nullspace, Left Nullspace

- Column space of A : $C(A)$

- Nullspace of A : $N(A)$

- Row space of A: $C(A^T)$
  - The row space of A is all combinations of the rows, or all combinations of the columns of A^T.
  - The rows of A, or the columns of A^T, span the row space.
  - If the rows of A are independent then they form a basis of the row space.

- Nullspace of A^T: $N(A^T)$
  - Usually referred to as the **Left Nullspace** of A
  - It consists of the vectors y for which $A^T y = 0$ or $y^T A = 0^T$

Dimensionality of the Four Fundamental Subspaces

- Assume that the matrix A is $m \times n$.
- The column space of A, $C(A)$, is a subset of R^m.
  - The dimension of $C(A)$ is $rank(A) = r$.
  - A basis of $C(A)$ consists of its pivot columns.
- The nullspace of A, $N(A)$, is a subset of R^n.
  - The dimension of $N(A)$ is $n - r$.
  - A basis of $N(A)$ consists of the special solutions.
- The row space of A, $C(A^T)$, is a subset of R^n.
  - The dimension of $C(A^T)$ is $rank(A) = r$.
  - A basis of $C(A^T)$ consists of the first r rows of the RREF matrix R. Note that the last $m - r$ rows of R are zero.
- The nullspace of A^T, $N(A^T)$, is a subset of R^m.
  - The dimension of $N(A^T)$ is $m - r$.
  - **A basis of $N(\mathrm{A}^T)$ consists of the last $m - r$ rows of E.**
  - **NOTE THAT: $E_{m\times m}[A_{m\times n}\ I_{m\times m}] = [R_{m\times n}\ E_{m\times m}]$ (please see Prob. 1, Sheet 2)**