Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

Lecture 3, Friday 14th October 2016

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Mathematics for Signals and Systems

In this set of lectures we will tackle the following problems:

- Column Space, Row Space and Rank of a matrix
- Vector Spaces and Subspaces
- Column Spaces and Nullspaces
- Solving $Ax = 0$
- Pivot / Free Variables
- Special Solutions

Background
Column Space, Row Space and Rank of a matrix

- In linear algebra, we define the **column space** $C(A)$ of a matrix A (sometimes called the **range** of a matrix) as the set of all possible linear combinations of its column vectors.
- Consider a matrix A of size $m \times n$. Its columns are $m -$ dimensional vectors. Therefore, its column space is a linear subspace of the $m\ -$ dimensional plane R^m.
- The dimension of the column space of a matrix A is called the **rank** of the matrix.
- We define the **row space** $R(A)$ of a matrix A as the set of all possible linear combinations of its row vectors.
- Consider a matrix A of size $m \times n$. Its rows are $n\ -$ dimensional vectors. Therefore, its row space is a linear subspace of the $n\ -$ dimensional plane R^n.
- **The column and row space of a matrix are always of the same dimension!**
- Therefore, the dimension of the row space of a matrix A also defines the **rank** of the matrix A.
- **Based on the above, the rank of a matrix is at most $\min(m, n)$!!!**

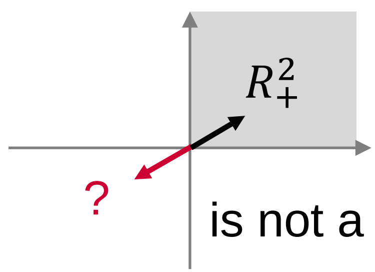
Rank: What you know so far

To summarise the previous material, let A be an $m \times n$ matrix. So far you know that:

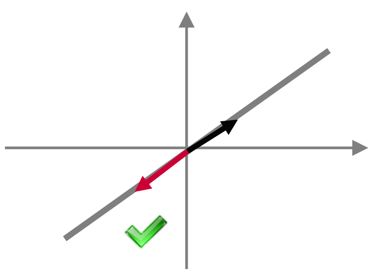
- $rank(A) = \dim\big(R(A)\big) = \dim\big(C(A)\big)$
- $rank(A)$ = the maximum number of linearly independent rows or columns of A.
- $rank(A) \leq \min(m, n)$
- **Keep in mind and don’t forget that the column and row space of a matrix are of the same dimension, but they are different spaces!**

Mathematics for Signals and Systems: Vector Spaces

- An N –dimensional space in which we can define specific vector operations is called **vector space**.
- For example, R^2 ($x - y$ plane) is a vector space where operations on 2-dimensional vectors can be defined.
- Note that all vectors with two real components are included in R^2.
- A vector space must be closed under multiplication and addition. If it is not, then it is NOT a vector space! This means that:
  - The product of a vector with a real number has to be in the vector space.
  - Any linear combination of vectors in the vector space has to be in the vector space.



is not a vector space



is a vector space (subspace of R^2)

Mathematics for Signals and Systems: Subspaces

- Examples of vector spaces which are subspaces of R^2:
  - All of R^2 (plane).
  - All lines that go through the origin (line).
  - Zero vector only (point).

- Examples of vector spaces which are subspaces of R^3:
  - All of R^3.
  - All planes that go through the origin (R^2 planes).
  - All lines through the origin (lines).
  - Zero vector only (point).

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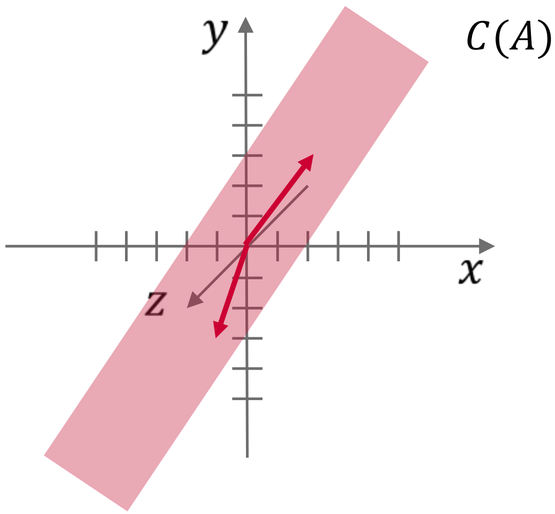
Mathematics for Signals and Systems: Column Space

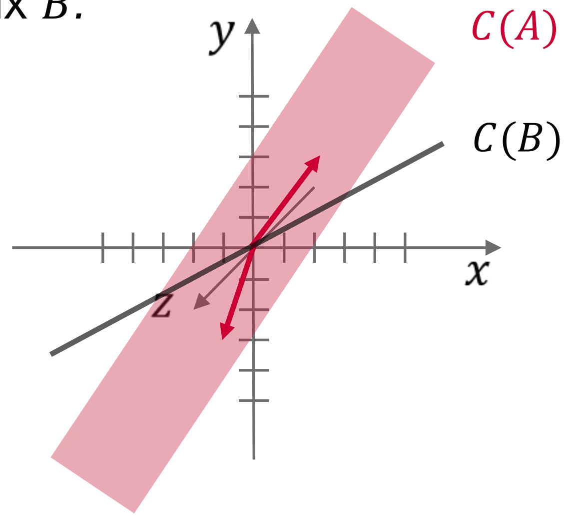
- Consider the columns of matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$.
- They are 3-dimensional vectors and therefore, they lie in R^3.
- Their linear combinations form a subspace of R^3.
- This subspace of R^3 is called the **Column Space of *A*** and it is denoted by $C(A)$.
- In that particular example, matrix A has two independent columns that lie in R^3.
- The column space of matrix A is a two dimensional plane (subspace) that goes through the origin.



Mathematics for Signals and Systems: Column Space cont.

- Consider the column spaces of matrix A and matrix B.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$



- The union of $C(A)$ and $C(B)$ is not a subspace.
- Linear combination of columns A and B do not necessarily lie in the union.
- The intersection of $C(A)$ and $C(B)$ is a subspace. This is because intersection is the zero vector which is a subspace.
- In general, **the intersection of subspaces is always a subspace**.

Mathematics for Signals and Systems: Column Space cont.

- A system of linear equations doesn't always have a solution for every b.

$$Ax = b$$

- Consider the example where we have 4 equations with 3 unknowns:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b$$

- If b is a linear combination of the columns of A then the system has a solution.
- The column space $C(A)$ contains, by definition, all the linear combinations Ax of the columns of A.
- Therefore, we can solve $Ax = b$ exactly only when b is in the column space of $C(A)$.

Mathematics for Signals and Systems: Column Space cont.

- Notice that in the previous example, column 3 of A is a linear combination of the other two columns.

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b$$

- The first two columns are independent.
- Independent columns are also known as pivot columns.
- The column space $C(A)$ of matrix A is a two dimensional plane which is a subspace of R^4.
- The number of pivot (independent) columns defines the column space of a matrix.

Mathematics for Signals and Systems: Nullspace

- The nullspace of A, denoted as $N(A)$, contains all possible solutions of the system:

$$Ax = 0$$

- Consider the example:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- In this example $C(A)$ is a subspace of R^4 and $N(A)$ is a subspace of R^3.
- We can see that the solutions of $Ax = 0$ have the form:

$$c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

- So the nullspace of A is a line (as defined above) which is a subspace of R^3.

Mathematics for Signals and Systems: Nullspace cont.

- The solutions to $Ax = 0$ always form a subspace.

- This is because any linear combination of the solutions to $Ax = 0$ is also a solution since if $Av = 0$ and $Aw = 0$ then $A(v + w) = Av + Aw = 0$.

- Also any multiple of the solutions to $Ax = 0$ is also a solution since if $Av = 0$ then $A(cv) = cAv = 0$.

Mathematics for Signals and Systems
Column Spaces and Nullspaces

- Consider again the system $Ax = b$ where b is non-zero:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- The solutions to $Ax = b$ where b is non-zero do not form a subspace.

- The above statement can be easily verified by the fact that the zero vector is not a solution, and therefore, the solutions cannot form a subspace.

- In other words, the solutions to $Ax = b$ lie in a plane or line that doesn't go through the origin, and hence, they don't form a subspace.

Mathematics for Signals and Systems
Column Spaces and Nullspaces

- The column space $C(A)$ contains all the linear combinations of the form Ax.
- The nullspace $N(A)$ contains all the solutions to the system $Ax = 0$.
- Lets see how we describe the column space and nullspace.
- We will describe / compute the nullspace of following matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

- We will perform elimination on this rectangular matrix.
- Note that a number of columns and rows are not independent. For example the second column is obtained from the first column if the later is multiplied by 2.
- This will become apparent during elimination.

Mathematics for Signals and Systems
Computing the Nullspace

- We will solve the system $Ax = 0$ by elimination.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

- During elimination the nullspace remains unchanged, since the solution to $Ax = 0$ does not change by elimination.
- The first two steps of elimination yield the matrix below right.
- Note that we can't find a pivot in the second column, meaning that the second column is not independent (depends on the previous column).

$$\begin{matrix} [2]-2[1] \\ [3]-3[1] \end{matrix} \quad \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Mathematics for Signals and Systems
Computing the Nullspace

- We ignore the fact that we can't find a pivot in the second column and we continue the elimination in the third column.
- We also notice that the last column doesn't have a pivot and it also depends on the previous columns.

$$\begin{matrix}1 & 2 & 2 & 2\\ 2 & 4 & 6 & 8\\ 3 & 6 & 8 & 10\end{matrix} \quad \begin{matrix}[2]-2[1]\\ [3]-3[1]\end{matrix} \quad \rightarrow \quad \begin{matrix}1 & 2 & 2 & 2\\ 0 & 0 & 2 & 4\\ 0 & 0 & 2 & 4\end{matrix} \quad [3]-[2] \quad \rightarrow \quad \begin{matrix}\boxed{1} & 2 & 2 & 2\\ 0 & 0 & \boxed{2} & 4\\ 0 & 0 & 0 & 0\end{matrix}$$

- Therefore, in this case we only have 2 pivots, signifying the number of independent columns.
- The number of pivots is called the rank of the matrix.
- Therefore, in this particular example $rank(A) = 2$.
- Note that in the rectangular **(non-square)** case the resulting matrix u is not really an upper triangular, but it is in the so-called **Echelon** (staircase) form.
- In order to identify the nullspace we need to describe the solutions of $Ax = 0$.

Mathematics for Signals and Systems
Computing the Nullspace

- By applying elimination we obtained the upper triangular matrix u.

$$u = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The matrix u contains two **pivot columns** shown in **red** and two **free columns** shown in **blue**.
- The free columns represent free variables, i.e., variables that we can assign any values to them.
- We obtain the null space of $Ax = 0$, by solving the system $ux = 0$.

Mathematics for Signals and Systems
Computing the Nullspace

- In order to calculate the null space we need to solve the system $Ax = 0$.
- The system $Ax = 0$ is equivalent to $ux = 0$ which can be written as:

$$ux = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

- Using row formulation we obtain:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \Rightarrow x_1 + 2x_2 - 4x_4 + 2x_4 = 0 \Rightarrow x_1 = -2x_2 + 2x_4$$

$$2x_3 + 4x_4 = 0 \Rightarrow x_3 = -2x_4$$

- The solution of the above system is of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

- x_2 and x_4 can take any values (free variables).

Mathematics for Signals and Systems
Computing the Nullspace

- By assigning the value of 1 to a particular free variable and the value of 0 to the rest of the free variables we obtain a so called **special solution**.
- First Special Solution is obtained for $x_2 = 1,\ x_4 = 0$ and is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Second Special Solution is obtained for $x_2 = 0,\ x_4 = 1$ and is

$$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

- **The null space is the linear combination of the special solutions:**

$$c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Mathematics for Signals and Systems
Computing the Nullspace-Summary

- We have seen that the matrix u contains two pivot and two free columns.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- In general if an $m \times n$ matrix has rank r, it has r pivot variables and $n - r$ free variables.
- The matrix has r independent columns and $n - r$ dependent columns.
- We choose freely $n - r$ variables.
- By assigning the value of 1 to a particular free variable and the value of 0 to the rest of the free variables we obtain a so called special solution.
- There are obviously $n - r$ special solutions.
- The linear combinations of these $n - r$ special solutions constitute the nullspace $Ax = 0$.
- The column space of A has dimension r and the nullspace has dimension $n - r$.

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Mathematics for Signals and Systems
Computing the Nullspace

- Now lets continue with elimination upwards to make the matrix even more “sparse”.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}, u = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Matrix u is in an **echelon** form.
- We notice that it has a row of zeros.
- Therefore, elimination revealed the fact that the third row of A is a linear combination of rows one and two.
- We continue with eliminations upwards to get zeros above (and below) the pivots.
- Finally we divide the second row by 2 to get one at the pivot positions.

$$[1]-[2] \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [2]/2 \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Mathematics for Signals and Systems
Computing the Nullspace

- The matrix R is said to have a **reduced row echelon form**, it has unit pivots and zeros above and below the pivots.

$$R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- In MATLAB we can calculate the reduced row echelon form of matrix A with the command:

$$R = rref(A)$$

- Notice the identity matrix which occupies the pivot rows and columns and represents the independent part of matrix A.
- The rest of the matrix contains the free columns.
- **Homework:** Find the nullspace of A^T.