Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

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Positive definite matrices: a bit more on this!

- If a matrix A is positive-definite, its inverse A^{-1} it also positive definite. This comes from the fact that the eigenvalues of the inverse of a matrix are equal to the inverses of the eigenvalues of the original matrix.

- If matrices A and B are positive definite, then their sum is positive definite. This comes from the fact $x^T(A+B)x = x^TAx + x^TBx > 0$. The same comment holds for positive semi-definiteness.

- Consider the matrix A of size $m \times n$ (rectangular, not square). In that case we are interested in the matrix A^TA which is square.

- Is A^TA positive definite?

Positive definite matrices: a bit more on this!

- Is $A^T A$ positive definite?

- $x^T A^T A x = (Ax)^T A x = ||Ax||^2$

- In order for $||Ax||^2 > 0$ for every $x \neq 0$, the null space of A must be zero.

- In case of A being a rectangular matrix of size $m \times n$ with $m > n$, the rank of A must be n.

Similar matrices

- Consider two square matrices A and B.
- Suppose that for some invertible matrix M the relationship $B = M^{-1}AM$ holds. In that case we say that A and B are similar matrices.
- **Example:** Consider a matrix A which has a full set of eigenvectors. In that case $S^{-1}AS = \Lambda$. Based on the above A is similar to Λ.
- **Similar matrices have the same eigenvalues! Matrices with identical eigenvalues are not necessarily similar!** There are different "families" of matrices with the same eigenvalues.
- Consider the matrix A with eigenvalues λ and corresponding eigenvectors x and the matrix $B = M^{-1}AM$.

  We have $Ax = \lambda x \Rightarrow AMM^{-1}x = \lambda x \Rightarrow M^{-1}AMM^{-1}x = \lambda M^{-1}x$

  $$BM^{-1}x = \lambda M^{-1}x$$

  Therefore, λ is also an eigenvalue of B with corresponding eigenvector $M^{-1}x$.

Similar matrices in the “bad” case of repeated eigenvalues

- Consider the “families” of matrices with repeated eigenvalues.
- **Example:** Lets take the 2×2 size matrices with eigenvalues $\lambda_1 = \lambda_2 = 4$.
  - The following two matrices

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I, \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

    have eigenvalues $4,4$ but they belong to different families, i.e., they are not similar.
  - There are __**two**__ families of matrices with eigenvalues $4,4$.
  - The matrix $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ is a loner. The only matrix similar to it, is itself!
  - The “big” family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ and any matrix of the form $\begin{bmatrix} 4 & a \\ 0 & 4 \end{bmatrix}$, $a \neq 0$. These matrices are not diagonalizable since they only have one non-zero eigenvector.

Similar matrices in the “bad” case of repeated eigenvalues

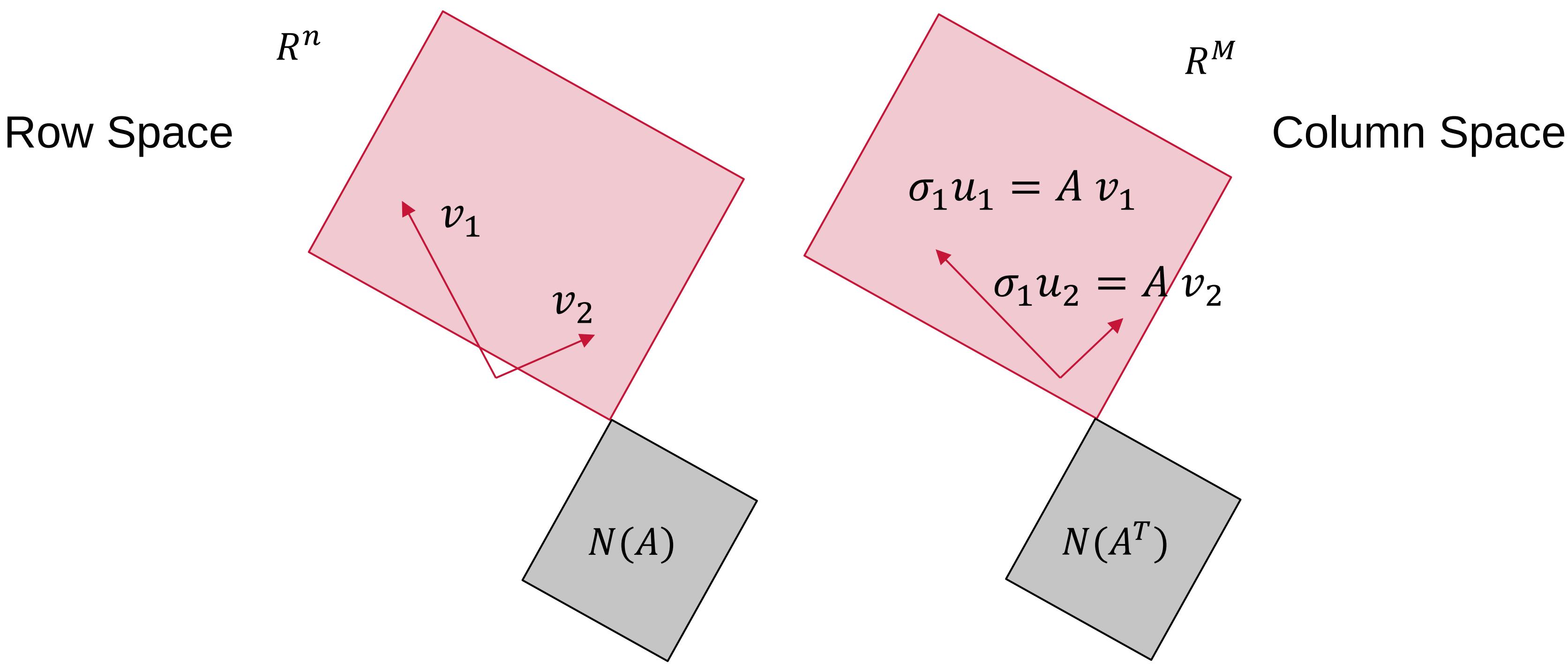
- Lets find more matrices of the family of $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$.
- Any matrix with trace 8 and determinant 16 belongs to that family.
- Examples are $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$.
- Similar matrices with repeated eigenvalues have identical eigenvalues and same number of independent eigenvectors. The reverse is not true.

Singular Value Decomposition (SVD)

- **Any** matrix A can be factorized as $A = U\,\Sigma\,V^T$.
  - » U is an orthogonal matrix with columns u
  - » Σ is a diagonal matrix
  - » V is an orthogonal matrix with columns v
- U is in general different to V.
- When A is a square invertible matrix then $A = S\,\Lambda\,S^{-1}$.
- When A is a symmetric positive definite matrix, the eigenvectors of S are orthogonal, so $A = Q\,\Lambda\,Q^T$.
- Therefore, for symmetric positive definite matrices SVD is effectively an eigenvector decomposition $U = Q = V$ and $\Lambda = \Sigma \;\; (\text{positive})$.

Singular Value Decomposition (SVD)

- With SVD an orthogonal basis in the row space, which is given by the columns of v, is mapped by matrix A to an orthogonal basis in the column space given by the columns of u. This comes from $AV = U\,\Sigma$.



Singular Value Decomposition (SVD)

- In matrix form the mapping between the row and column space that the SVD achieves can be written as:

$$A\,[v_1 \quad \dots \quad v_r] = [u_1 \quad \dots \quad u_r]\begin{bmatrix}\sigma_1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r\end{bmatrix} \text{ or } A\,V = U\,\Sigma.$$

- So the goal is to find an orthonormal basis (V) of the row space and an orthonormal basis (U) of the column space that diagonalize the matrix A to Σ.
- In the generic case the basis of V would be different to the basis of U.
- Note that if A is singular, the null space of A is not empty. Then the SVD is written as:

$$A\,[v_1 \quad \dots \quad v_r \quad v_{r+1} \quad \dots \quad v_n] = [u_1 \quad \dots \quad u_r \quad u_{r+1} \quad \dots \quad u_n]\begin{bmatrix}\sigma_1 & \dots & 0 & 0\\ \vdots & \sigma_r & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Singular Value Decomposition (SVD)

- The following relationships hold:

$$AV = U\Sigma$$
$$A = U\Sigma V^{-1} = U\Sigma V^T$$

- The matrix $A^T A$ is therefore

$$A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T \text{ with}$$
$$\Sigma = \begin{bmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n^2 \end{bmatrix}$$

- Therefore, the above expression is the eigenvector decomposition of $A^T A$.
- Similarly, the eigenvector decomposition of AA^T is:

$$AA^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T$$

- So we can determine all the factors of SVD by the eigenvalue decompositions of matrices $A^T A$ and $A A^T$.

Singular Value Decomposition (SVD)

- Example: $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ and $A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$
- The eigenvalues of $A^T A$ are 32 and 18.
- The eigenvectors of $A^T A$ are $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and

$$A^T A = V \Sigma^2 V^T$$

- Similarly $AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$
- Therefore, the eigenvectors of AA^T are $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $AA^T = U \Sigma^2 U^T$.
- $Note\ that$: $eig(AB) = eig(BA)$

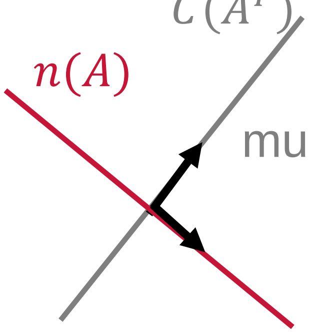
Singular Value Decomposition (SVD)

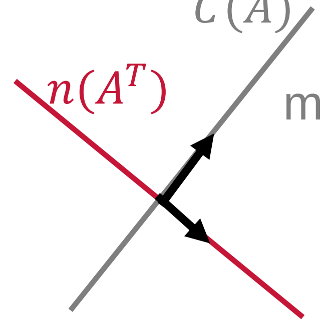
- Therefore, the SVD of $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ is:

$$A = U\,\Sigma\,V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

Singular Value Decomposition (SVD)

- **Example:** The matrix A is singular $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$





- The eigenvalues of $A^T A = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$ are 0 and 125.

$$A = U\,\Sigma\,V^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

Singular Value Decomposition (SVD)

- Orthonormal basis for row space: $v_1 \quad \cdots \quad v_r$
- Orthonormal basis for column space: $u_1 \quad \cdots \quad u_r$
- Orthonormal basis for null space: $v_{r+1} \quad \cdots \quad v_n$
- Orthonormal basis for null space of A^T: $u_{r+1} \quad \cdots \quad u_n$

These bases make matrix A diagonal $Av_i = \sigma_i u_i$