Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

Lecture 18, Friday 18th November 2016

DR TANIA STATHAKI

READER (ASSOCIATE PROFFESOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

Mathematics for Signals and Systems

In this set of lectures we will talk about two applications:

- Linear Transformations
- Summary of Decompositions and Matrices

Linear transformations

- Consider the parameters/functions/vectors/other mathematical quantities denoted by u and v.

- A transformation is an operator applied on the above quantities, i.e., $T(u), T(v)$.

- A linear transformation possesses the following two properties:
  - $T(u+v) = T(u) + T(v)$
  - $T(cv) = cT(v)$ where c is a scalar.

- By grouping the above two conditions we get

$$T(c_1 u + c_2 v) = c_1 T(u) + c_2 T(v)$$

- The zero vector in a linear transformation is always mapped to zero.

Examples of transformations

- Is the transformation $T\colon R^2 \to R^2$, which carries out projection of any vector of the 2-D plane on a specific straight line, a linear transformation?

- Is the transformation $T\colon R^2 \to R^2$, which shifts the entire plane by a vector v_0, a linear transformation?

- Is the transformation $T\colon R^3 \to R$, which takes as input a vector and produces as output its length, a linear transformation?

- Is the transformation $T\colon R^2 \to R^2$, which rotates a vector by 45^{o} a linear transformation?

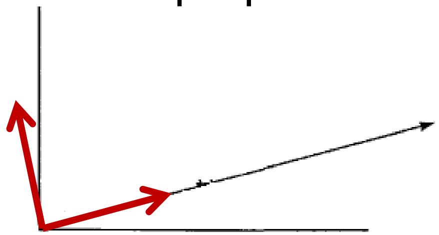
- Is the transformation $T(v) = Av$, where A is a matrix, a linear transformation?

Examples of transformations

- Consider a transformation $T\colon R^3 \rightarrow R^2$.
- In case $T(v) = Av$, then A is a matrix of size 2×3.
- If we know the outputs of the transformation if applied on a set of vectors $v_1, v_2, \dots, v_n$ which form a basis of some space, then we know the output to any vector that belongs to that space.
- **Recall: The coordinates of a system are based on its basis.**
- Most of the time when we talk about coordinates we think about the "standard" basis, which consists of the rows (columns) of the identity matrix.
- Another popular basis consists of the eigenvectors of a matrix.

Examples of transformations: Projections

- Consider the matrix A that represents a linear transformation T.
- Most of the times the required transformation is of the form $T\colon R^n \to R^m$.
- I need to choose two bases, one for R^n, denoted by $v_1, v_2, \dots, v_n$ and one for R^m denoted by $w_1, w_2, \dots, w_m$.
- I am looking for a transformation that if applied on a vector described with the input coordinates produces the output co-ordinates.
- Consider R^2 and the transformation which projects any vector on the line shown on the figure below.
- I consider as basis for R^2 the vectors shown with red below and not the "standard" vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.
- One of the basis vectors lies on the required line and the other is perpendicular to the former.



Examples of transformations: Projections cont.

- I consider as basis for R^2 the vectors shown with red below both before and after the transformation.
- Any vector v in R^2 can be written as $v = c_1 v_1 + c_1 v_2$.
- We are looking for $T(\cdot)$ such that

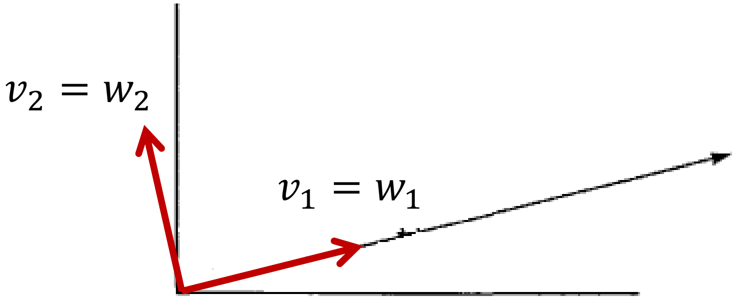
  $T(v_1) = v_1$ and $T(v_2) = 0$.

- Furthermore,

  $T(v) = c_1 T(v_1) + c_2 T(v_2) = c_1 v_1$

  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$
- The matrix in that case is Λ.

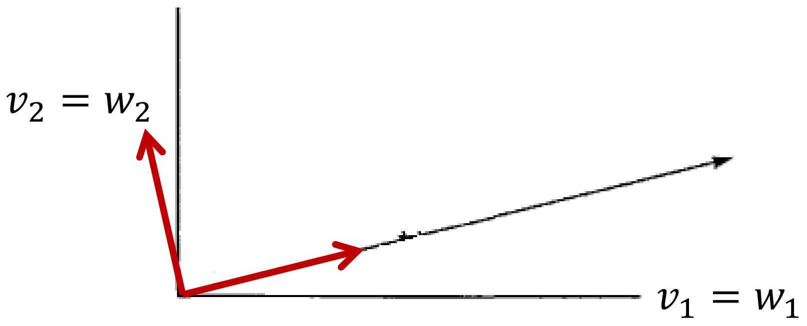
  This is the "good" matrix.



Imperial College London

Examples of transformations: Projections cont.

- I now consider as basis for R^2 the "standard" basis.
- $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Consider projections on to 45^{o} line.
- In this example the required matrix is
  $P = \frac{aa^T}{a^T a} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$
- Here we didn't choose the "best" basis, we chose the "handiest" basis.



Examples of transformations: Derivative of a function

- Consider a linear transformation that takes the derivative of a function. (The derivative is a linear transformation!)
- $T = \frac{d(\cdot)}{dx}$
- Consider input $c_1 + c_2x + c_3x^2$. Basis consists of the functions $1, x, x^2$.
- The output should be $c_2 + 2c_3x$. Basis consists of the functions $1, x$.
- I am looking for a matrix A such that $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$.
- This is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Summary of Decompositions: LU Decomposition

- What is it?
  - $A = LU,\ A\ is$ a square matrix.
  - L a lower triangular matrix with 1s on the diagonal.
  - U an upper triangular matrix with the pivots of A on the diagonal.

- When does it exist?
  - If the matrix is invertible (the determinant is not 0), then a pure LU decomposition exists only if the **leading principal minors** are not 0. The leading principal minors are the "north-west" determinants.

    Example:

    The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an LU although it is invertible.
  - If the matrix is not invertible (the determinant is 0), then we can't know if there is a pure LU decomposition.

- LU decomposition works with rectangular matrices as well, with slight modifications/extensions.

LDU Decomposition

- What is it?
  - $A = LDU, A$ is a square matrix.
  - L a lower triangular matrix with 1s on the diagonal.
  - D a diagonal matrix with the pivots of A across the diagonal.
  - U an upper triangular matrix with 1s on the diagonal.

- When does it exist?
  - The same requirements as in pure LU decomposition.

    Example:

$$A = LU = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

    The above can be written as:

$$A = LDU = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

- LDU decomposition works with rectangular matrices as well, with slight modifications/extensions.

Imperial College London

A=ER

- What is it?
  - $A = ER$
  - A is any matrix of dimension $m \times n$.
  - E a square invertible matrix of dimension $m \times m$.
  - $R = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix}$
  - F is a random matrix.
  - r is the rank of A.

- When does it exist?
  - Always.

A=QR for square matrices

- What is it?
  - $A = QR$
  - A is any matrix of dimension $n \times n$.
  - Q a square invertible matrix of dimension $n \times n$ with orthogonal columns.
  - The columns of Q are derived from the columns of A.
  - If A is not invertible the first r columns of Q are derived from the independent columns of A. r is the rank of A. The last $(n - r)$ columns of Q are chosen to be orthogonal to the first r ones.
  - R is upper triangular if A is invertible. If A is NOT invertible the last $(n - r)$ rows of R are filled with zeros. (Look at Problem Sheet 6, Question 6, Matrix B).

- When does it exist?
  - Always.

- QR decomposition exists also for rectangular matrices. I haven't taught this case.

$A = S\Lambda S^{-1}$

- What is it?
  - $A = S\Lambda S^{-1}$
  - A is a square invertible matrix of dimension $n \times n$.

- When does it exist?
  - When A has n linearly independent eigenvectors.

$A = Q\Lambda Q^T$

- What is it?
  - $A = Q\Lambda Q^T$
  - A is a real, symmetric matrix of dimension $n \times n$.
  - The above decomposition is the so called **Spectral Theorem**.

- When does it exist?
  - When A is real and symmetric.

The Singular Value Decomposition $A = U\Sigma V^T$

- What is it?
  - $A = U\Sigma V^T$
  - A is any matrix of dimension $m \times n$.
  - U is an orthogonal matrix of dimension $m \times m$ whose columns are the eigenvectors of matrix AA^T.
  - Σ is a singular value matrix, with the singular values of A in its main diagonal.
  - V is an orthogonal matrix of dimension $n \times n$ whose columns are the eigenvectors of matrix $A^T A$.
  - The squares of the singular values of A are the eigenvalues of both AA^T and $A^T A$.

- When does it exist?
  - Always.

Summary of matrices

- Projection matrix P onto subspace S.
  - $p = Pb$ is the closest point to b in S.
  - $P^2 = P = P^T$.
  - Condition $P^2 = P$ is sufficient to characterise a matrix as a projection matrix.
  - Eigenvalues: 0 or 1.
  - Eigenvectors are in S or $S^\perp$.
  - If the columns of matrix A are a basis for S, then $P = A(A^T A)^{-1} A^T$.

- Orthogonal matrix Q.
  - Square matrix with orthonormal columns.
  - $Q^{-1} = Q^T$
  - Eigenvalues: $|\lambda| = 1$.
  - Eigenvectors are orthogonal.
  - Preserves length and angles, i.e., $\|Qx\| = \|x\|$.

Imperial College London

Also recall

- Symmetric (or Hermitian) matrices (real eigenvalues).
- Positive definite and positive semi-definite matrices. Their eigenvalues are positive or non-negative respectively.
- Matrices $A^T A$ and AA^T.
  - They have identical eigenvalues.
  - Their eigenvalues are non-negative.
- Square matrices.
- Rectangular matrices.