Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

Lecture 16

DR TANIA STATHAKI

READER (ASSOCIATE PROFFESOR) IN SIGNAL PROCESSING

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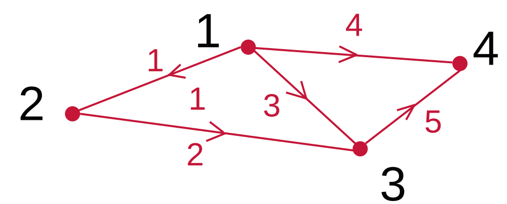
In this set of lectures we will talk about…

- an application of linear system theory: graphs and networks
- orthogonal spaces

Graphs and networks: **incidence** matrix

- A **graph** is a set of nodes and edges denoted as

$$\text{graph} = \{\text{nodes}, \text{edges}\}$$



- The graph can be represented by a matrix (**incidence matrix**) where each row corresponds to an edge and each column corresponds to a node.
- The element $A_{ij} = 1$ if current flows towards node j accross edge i.
- The element $A_{ij} = -1$ if current flows away from node j accross edge i.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\} \text{1-2-3 loop} \\ 4 \\ 5 \end{matrix}$$

- A subgraph is formed by edges 1,2,3. This is a loop.
- Note that loops always correspond to linearly dependent rows.

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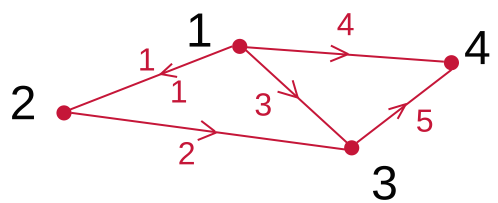
Graphs and networks: null space of incidence matrix

- The null space of matrix A is zero if the columns are independent. For the given example we have:

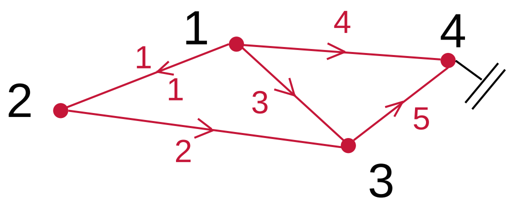
$$Ax = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = 0$$



- The vector x represents potentials at nodes (e.g. voltages).
- $x_i - x_j$ represents the difference in potential across certain edges.
- We see that the a solution of the above system is $x = [1 \quad 1 \quad 1 \quad 1]^T$.
- The null space is formed by vectors $c[1 \quad 1 \quad 1 \quad 1]^T$ and $\dim\big(N(A)\big) = 1$.
- The solution to the above system is obtained subject to a scalar c.
- Since $n = 4$ and and $\dim\big(N(A)\big) = 1$, we get $\mathrm{rank}(A) = 3$.

Graphs and networks: null space of transpose of incidence matrix

- By fixing the potential at node one to 0 we remove a column and we solve for the remaining potentials.
- Let us consider the equation



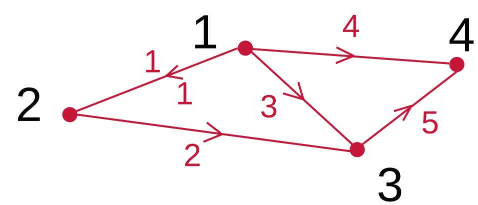
$$A^T y = 0 \Rightarrow \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = 0$$

- The vector y represents currents across the edges.
- The equation $A^T y = 0$ represents Kirchoff's law.
- (Note that there is a matrix C that connects potential differences and current at the edges, and represent Ohm's law: $y = Ce$).

Graphs and networks: Kirchoff's law

- The equation $A^T y = 0$ is Kirchoff's law.

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = 0$$



- The first equation refers to node one and indicates that the net current flow is zero. Similarly we get:

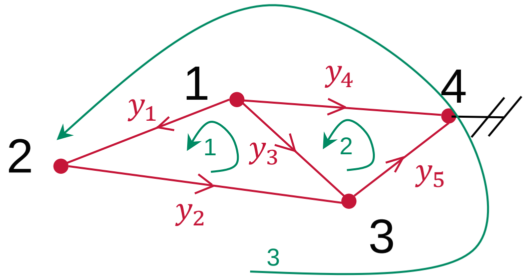
$$-y_1 - y_3 - y_4 = 0$$

$$y_1 - y_2 = 0$$

$$y_2 + y_3 - y_5 = 0$$

$$y_4 + y_5 = 0$$

Graphs and networks: Kirchoff’s law



- Three solution vectors that satisfy Kirchoff’s law represent total current running across the three possible loops.

$$-y_1 - y_3 - y_4 = 0$$

$$y_1 - y_2 = 0$$

$$y_2 + y_3 - y_5 = 0$$

$$y_4 + y_5 = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

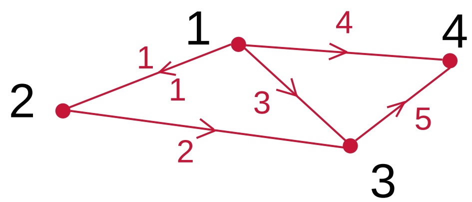
- We can see the third solution (current running across loop 3) is not independent from the first two solutions.
- The null space of A^T is two dimensional, which is the same as the number of loops.

$$\dim(N(A^T)) = 2$$

Graphs and networks: row space of incidence matrix

- Consider the columns space of A^T which is the row space of A.

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$



- The pivot columns of A^T are the first, second and the fourth, that form a graph without loops. This graph is called a **tree**.

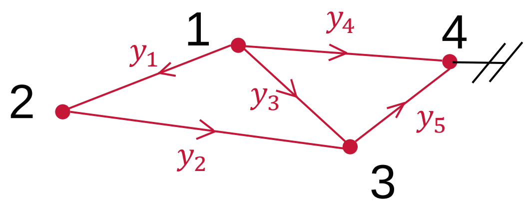
$$\dim(N(A^T)) = m - r$$

$$\#\text{loops} = \#\text{edges} - (\#\text{nodes} - 1)$$

$$\#\text{nodes} - \#\text{edges} + \#\text{loops} = 1 \qquad \text{(Euler's formula)}$$

Graphs and networks

- Summarizing all the equation



Potential differences: $e = Ax$
Ohm's Law: $y = Ce$
Kirchoff's Current Law: $A^T y = 0$

- The above three equations can be merged in a single equation as follows:

$$A^T C A x = 0$$

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Linear transformations

- Consider the parameters/functions/vectors/other mathematical quantities denoted by u and v.

- A transformation is an operator applied on the above quantities, i.e., $T(u), T(v)$.

- A linear transformation possesses the following two properties:
  - $T(u+v) = T(u) + T(v)$
  - $T(cv) = cT(v)$ where c is a scalar.

- By grouping the above two conditions we get

$$T(c_1 u + c_2 v) = c_1 T(u) + c_2 T(v)$$

- The zero vector in a linear transformation is always mapped to zero.

Examples of transformations

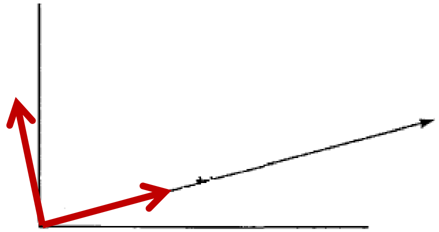
- Is the transformation $T\colon R^2 \rightarrow R^2$, which carries out projection of any vector of the 2-D plane on a specific straight line, a linear transformation?
- Is the transformation $T\colon R^2 \rightarrow R^2$, which shifts the entire plane by a vector v_0, a linear transformation?
- Is the transformation $T\colon R^3 \rightarrow R$, which takes as input a vector and produces as output its length, a linear transformation?
- Is the transformation $T\colon R^2 \rightarrow R^2$, which rotates a vector by 45^{o} a linear transformation?
- Is the transformation $T(v) = Av$, where A is a matrix, a linear transformation?

Examples of transformations

- Consider a transformation $T\colon R^3 \rightarrow R^2$.
- In case $T(v) = Av$, then A is a matrix of size 2×3.
- If we know the outputs of the transformation if applied on a set of vectors $v_1, v_2, \dots, v_n$ which form a basis of some space, then we know the output to any vector that belongs to that space.
- **Recall: The coordinates of a system are based on its basis!**
- Most of the time when we talk about coordinates we think about the "standard" basis, which consists of the rows (columns) of the identity matrix.
- Another popular basis consists of the eigenvectors of a matrix.

Examples of transformations: Projection

- Consider the matrix A that represents a linear transformation T.
- Most of the times the required transformation is of the form $T\colon\ R^n \rightarrow R^m$.
- I need to choose two bases, one for R^n, denoted by $v_1, v_2, \dots, v_n$ and one for R^m denoted by $w_1, w_2, \dots, w_m$.
- I am looking for a transformation that if applied on a vector described with the input coordinates produces the output co-ordinates.
- Consider R^2 and the transformation which projects any vector on the line shown on the figure below.
- I consider as basis for R^2 the vectors shown with red below and not the "standard" vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.
- On of the basis vectors lies on the required line and the other is perpendicular to the former.



Examples of transformations: Projection (cont)

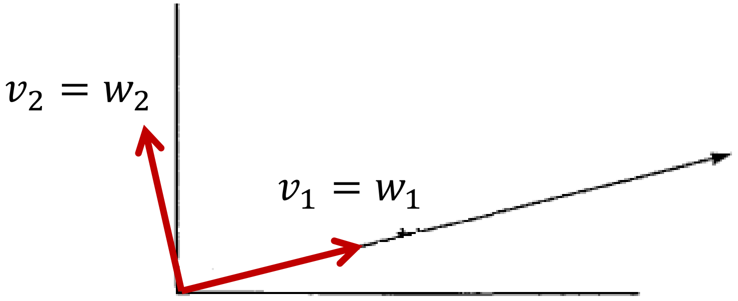
- I consider as basis for R^2 the vectors shown with red below both before and after the transformation.
- Any vector v in R^2 can be written as $v = c_1 v_1 + c_1 v_2$.
- We are looking for $T(\cdot)$ such that

  $T(v_1) = v_1$ and $T(v_2) = 0$.

  Furthermore,

  $T(v) = c_1 T(v_1) + c_1 T(v_2) = c_1 v_1$

  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

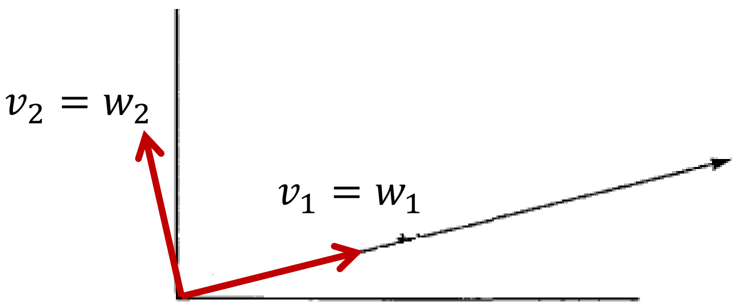


- The matrix in that case is Λ. This is the “good” matrix.

Examples of transformations: Projection (cont)

- I now consider as basis for R^2 the "standard" basis.
- $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Consider projections on to 45^o line.
- In this example the required matrix is

$$P = \frac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$



- Here we didn't choose the "best" basis, we chose the "handiest" basis.

Rule for finding matrix A

- Suppose we are given the bases $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_m$.
- How do I find the first column of A? The first column of A should tell me what happens to the first basis vector. Therefore, we apply $T(v_1)$. This should give

$$T(v_1) = a_{11}w_1 + a_{21}w_2 \dots a_{m1}w_m = \sum_{i=1}^{m} a_{i1}w_i$$

- We observe that $\{a_{i1}\}$ form the first column of the matrix A.
- In general $T(v_j) = a_{1j}w_1 + a_{2j}w_2 \dots a_{mj}w_m = \sum_{i=1}^{m} a_{ij}w_i$

Examples of transformations: Derivative of a function

- Consider a linear transformation that takes the derivative of a function. (The derivative is a linear transformation!)
- $T = \frac{d(\cdot)}{dx}$
- Consider input $c_1 + c_2 x + c_3 x^2$. Basis consists of the functions $1, x, x^2$.
- The output should be $c_2 + 2c_3 x$. Basis consists of the functions $1, x$.
- I am looking for a matrix A such that $A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$.

This is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Types of matrix inverses

- **2-sided inverse (or simply inverse)**

$r = m = n$ (full rank) $AA^{-1} = I = A^{-1}A$

- **Left inverse. (Note that a rectangular matrix cannot have a 2-sided inverse!)**

$r = n < m$

(full column rank)
independent columns
nullspace = $\{0\}$
0 or 1 solutions to $Ax = b$

$A^T A$
$n\ x\ n$
invertible

$$\underbrace{(A^T A)^{-1} A^T}_{A^{-1}_{left}} A = I$$

$$A^{-1}_{left}\ A\ =\ I$$
$n\ x\ m\quad m\ x\ n$

- **Right inverse**

$r = m < n$
$n - m$ free variables

(full row rank)
independent rows
$N(A^T) = \{0\}$
∞ solutions to $Ax = b$

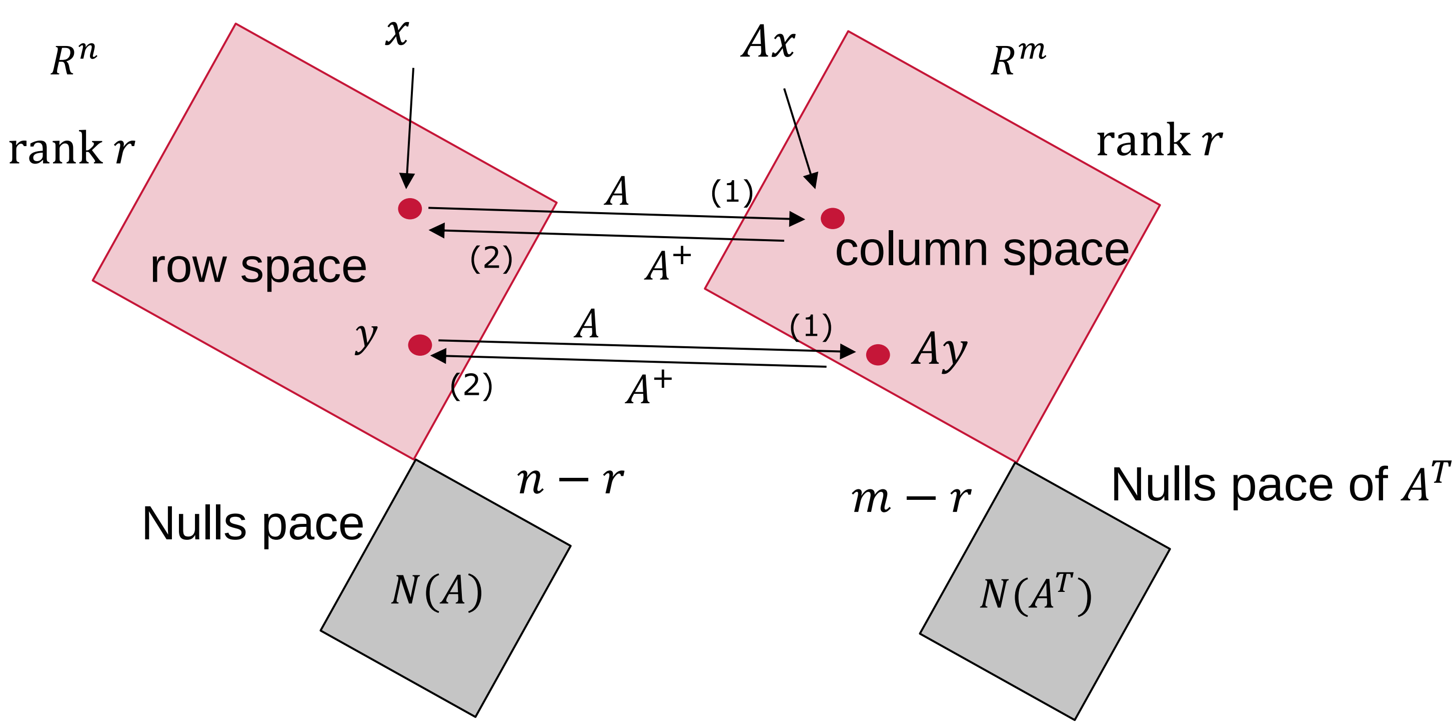
AA^T
$m\ x\ m$
invertible

$$A \underbrace{A^T (AA^T)^{-1}}_{A^{-1}_{right}} = I$$

$$A\ A^{-1}_{right}\ =\ I$$
$m\ x\ n\quad n\ x\ m$

Pseudo-inverse. The case for $r < m, r < n$

- The multiplication of a vector from the row space x with a matrix A gives a vector Ax in the column space (1)
- The multiplication of a vector from the column space Ax with the pseudo inverse of A (i.e. A^+) gives the vector $x = A^+Ax$ (2)



Pseudo-inverse

- If $x \neq y$ are different vectors in the row space then the vectors Ax, Ay are vectors in the column space. We can show that $Ax \neq Ay$.

  Proof

  Suppose $Ax = Ay$.

  Then $A(x - y) = 0$ is in the null space.

  But we know x, y and $x - y$ are in the row space.

  Therefore $x - y$ is the zero vector and $x = y$ so $Ax = Ay$.

- Therefore a matrix A is a mapping from row space to column space and vice-versa. For that particular mapping the inverse of A is denoted by A^{+} and is called pseudo-inverse.

Find the Pseudo-inverse

- How can we find the pseudo-inverse A^+
- Starting from SVD, $A = U\,\Sigma\,V^T$ with $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of size $m \times n$ and rank r.
- The pseudo-inverse is $A^+ = V\,\Sigma^+\,U^T$, $\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of size $n \times m$ and rank r.
- Note that $\Sigma\,\Sigma^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of size $m\ x\ m$ and is a projection matrix onto the column space.
- Note also that $\Sigma^+\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of size $n\ x\ n$ is a projection matrix onto the row space.
- $\Sigma\,\Sigma^+ \neq I \neq \Sigma^+\,\Sigma$.