Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

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Definition of Markov matrices and their properties

- Consider a matrix A with the following properties:
  - All entries are ≥ 0
  - The elements of each column add up to 1.
- This is called a **Markov matrix**.
- An example is

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 04 \end{bmatrix}$$

- When I square the matrix the above properties are still valid!
- The powers of the matrix are all Markov matrices.
- I am interested in the eigenvalues and eigenvectors of a Markov matrix.

Markov matrices and their eigenvalues

- $\lambda = 1$ is always an eigenvalue of a Markov matrix.
- The rest of the eigenvalues have magnitude $|\lambda_i| < 1$.
- Remember the system described by an equation of the form $u_k = A^k u_0$.
- We managed to write $u_k = A^k u_0 = c_1 {\lambda_1}^k x_1 + c_2 {\lambda_2}^k x_2 + \cdots$ where λ_i and x_i are the eigenvalues and eigenvectors of matrix A respectively.
- Note that the above relationship requires a complete set of eigenvectors.
- If $\lambda_1 = 1$ and $|\lambda_i| < 1, i > 1$ then the steady state of the system is $c_1 x_1$ (which is part of the initial condition u_0).
- Furthermore, the components of x_1 are positive or zero, i.e., the steady state is positive.

Markov matrices and their eigenvalues

- I would like to prove that $\lambda = 1$ is always an eigenvalue of a Markov matrix.
- Consider again the Markov matrix A.
- Since the elements of each column of A add up to 1, the elements of each column of $A - I$ should add up to zero.
- Therefore, the rows of $A - I$ should add up to zero.
- Therefore, matrix $A - I$ is singular, which yields $det(A - I) = 0$.
- From the above analysis, it is shown that $\lambda = 1$ is always an eigenvalue of a Markov matrix.
- Since $A - I$ is singular there is a vector x for which
$$(A - I)x = 0 \Rightarrow Ax = x$$
- A vector of the null space of $A - I$ is the eigenvector of A that corresponds to eigenvalue $\lambda = 1$.

Application of Markov matrices

- Consider again the system described by an equation of the form $u_k = A^k u_0$, where A is now a Markov matrix.
- I will use an example where A is a 2×2 matrix. Generally, an $n \times n$ Markov matrix is related to n “states”. This is a concept that we will develop shortly.
- Assume that A is a 2×2 matrix, and the 2 “states” are 2 UK cities.
- I take London and Oxford. I am interested in the population of the two cities and how it evolves.
- I assume that people who inhabit these two cities move between them only.
- $\begin{bmatrix} u_{\text{ox}} \\ u_{\text{lon}} \end{bmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{\text{ox}} \\ u_{\text{lon}} \end{bmatrix}_{t=k}$
- It is now obvious that the column elements are positive and also add up to 1 because they represent probabilities.

Application of Markov matrices (cont.)

- I assume that $\begin{bmatrix} u_{\text{ox}} \\ u_{\text{lon}} \end{bmatrix}_{t=k=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$
- $\begin{bmatrix} u_{\text{ox}} \\ u_{\text{lon}} \end{bmatrix}_{k=1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$
- **Question:** What is the population of the two cities after a long time?
- Consider the matrix $\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.7$.
  (Notice that the second eigenvalue is found by the trace of the matrix.)
- The eigenvectors of this matrix are $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} u_{\text{ox}} \\ u_{\text{lon}} \end{bmatrix}_k = c_1 {\lambda_1}^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 {\lambda_2}^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 0.7^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Application of Markov matrices (cont.)

- I find $c_1,\ c_2$ from the initial condition $\begin{bmatrix} u_{\mathrm{ox}} \\ u_{\mathrm{lon}} \end{bmatrix}_{t=k=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

  $\begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and therefore, $c_1 = \frac{1000}{3}$ and $c_2 = \frac{2000}{3}$

- Markov models facilitate the modeling of various real life engineering applications.

- An example is the modeling of the movement of people without gain or loss: total number of people is conserved!

Expansion with orthonormal basis

- Consider a set of orthonormal vectors $q_1, \dots, q_n$ which form a basis of the n –th dimensional space.
- Every vector v can be written as $v = x_1 q_1 + \cdots + x_n q_n$.
- The problem is to find x_i, for every i.
- The inner product ${q_1}^T \cdot v$ is given by
  ${q_1}^T \cdot v = {q_1}^T \cdot (x_1 q_1 + \cdots + x_n q_n) = x_1$
- The matrix Q contains the column vectors $q_i, i = 1, \dots, n$.

$$Qx = v \Rightarrow x = Q^{-1} v \text{ with } Q^{-1} = Q^T$$

Fourier Series

- The goal is to write a function $f(x)$ as follows:
  $$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$
- The cosine and sine signals are orthogonal signals.
- The difference of the above relationship to the previous one with vectors, is that this is infinite.
- The above relationship is the so called Fourier Series expansion.
- For the first time the vectors are replaced by functions.
- Orthogonal vectors v, w imply that $v^T w = v_1 w_1 + \cdots + v_n w_n = 0$.
- Orthogonal **continuous** functions f and g imply that
  $$f^T g = \int f(x) g(x) dx$$
- In the case of Fourier Series above I have $f^T g = \int_0^{2\pi} f(x) g(x) dx$.
- **Problem:** Find a_0, a_i, b_i