Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

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Determinant of a 2×2 matrix

- The goal is to find the determinant of a 2×2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ using the properties described previously.
- We know that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$.
- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} =$
  $0 + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + 0 = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc$
- As you can see, I break the determinant of a 2×2 random matrix into 4 determinants of simpler (permutation) matrices.
- I can implement the above analysis for 3×3 matrices.
- In the case of a 3×3 matrix I break the matrix into 27 determinants.
- And so on…

Determinant of a 2×2 matrix

- For the case of a 2×2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ we obtained:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0 + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + 0$$

- The determinants which survived have strictly one entry from each row and each column.

- The above is a universal conclusion!

Determinant of a 3 × 3 matrix

- For the case of a 3 × 3 matrix $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ we obtain:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \cdots =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}\,a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}\,a_{31} +$$

$$a_{13}a_{21}a_{32} - a_{13}a_{22}\,a_{31}$$

- As mentioned the determinants which survive have strictly one entry from each row and each column.

The "Big Formula" for the determinant

- For the case of a 2×2 matrix the determinant has 2 terms.
- For the case of a 3×3 matrix the determinant has 6 terms.
- For the case of a 4×4 matrix the determinant has 24 terms.
- For the case of a $n \times n$ matrix the determinant has $n!$ terms.
  - The elements from the first row can be chosen in n different ways.
  - The elements from the second row can be chosen in $(n-1)$ different ways
  - and so on…
- **Problem:** Find the determinant of the following matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The "Big Formula" for the determinant

- For the case of a $n \times n$ matrix the determinant has $n!$ terms.

$$det(A) = \sum_{n!\text{terms}} \pm a_{1a} a_{2b} a_{3c} \ldots a_{nz}$$

  - $a, b, c, \ldots, z$ are different columns
  - In the above summation, half of the terms have a plus and half of them have a minus sign.

The “Big Formula” for the determinant

- For the case of an $n\times n$ matrix, **cofactors** consist of a method which helps us to connect a determinant to determinants of smaller matrices.

$$det(A)=\sum_{n!\text{terms}} \pm a_{1a}a_{2b}a_{3c}\ldots a_{nz}$$

- Cofactors 3×3. Consider
$\det(A)=a_{11}(a_{22}a_{33}-a_{23}\,a_{32})-a_{12}(a_{21}a_{33}-a_{23}\,a_{31})$
$+a_{13}(a_{21}a_{32}-a_{22}\,a_{31})$

  $(a_{22}a_{33}-a_{23}\,a_{32})$ is the determinant of a 2×2 matrix which is a sub-matrix of the original matrix. We denote $C_{11}=a_{22}a_{33}-a_{23}\,a_{32}$.

Cofactors

- The cofactor of element a_{ij} is defined as follows:

$$C_{ij} = \pm\det\left[(n-1)\times(n-1) \text{ matrix } A_{ij}\right]$$

  - A_{ij} is the $(n-1)\times(n-1)$ matrix that is obtained from the original matrix A if row i and column j are eliminated.
  - We keep the $+$ if $(i+j)$ is even.
  - We keep the $-$ if $(i+j)$ is odd.

- Cofactor formula along row 1:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}\,C_{1n}$$

- Cofactor formula along any row or column can be used for the final estimation of the determinant.
- We define a matrix C with elements C_{ij}.

Estimation of the inverse A^{-1} using cofactors

- For a 2×2 matrix it is quite easy to show that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Big formula for A^{-1}

$$A^{-1} = \frac{1}{det(A)} C^T$$
$$AC^T = det(A) \cdot I$$

- C_{ij} is the cofactor of a_{ij} . For a matrix A of size $n \times n$, C_{ij} is always a product of $(n-1)$ entries.
- In general

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \det(A) \cdot I$$

Solve $Ax = b$

- The solution can be now obtained from

$$x = A^{-1}b = \frac{1}{det(A)} C^T b$$

- Cramer's rule:
  - First component of the answer $x_1 = \frac{\det(B_1)}{det(A)}$. Then $x_2 = \frac{\det(B_2)}{det(A)}$ and so on.
  - What are these matrices B_i?

$$B_1 = [b \,\vdots\, \text{ last } (n-1) \text{ columns of } A]$$

  - B_1 is obtained by A if we replace the first column with b. B_i is obtained by A if we replace the i column with b.
  - Is this rule "good" in practice? We must find $(n+1)$ determinants. **This will take forever!** But…
  - Having a formula allows you to have algebra instead of algorithms!

$\det(A)$ = volume of a box

- Take A to be a matrix of size 3×3.
- Then we can prove that $\det(A)$ is the volume of a 3D box.
- Look at the three-dimensional box (parallelepiped) formed from the three rows of A.
- It is proven that $\text{abs}(|A|)$=volume of the box!