Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

Class 2, Tuesday 2nd November 2014

DR TANIA STATHAKI

READER (ASSOCIATE PROFFESOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

Problem: Block elimination

- Consider the set of two systems of equations:

$$Ax + By = a$$
$$Cx + Dy = b$$

where A, B, C, D are square matrices of size $n \times n$ and a, b are column vectors of size $n \times 1$.

- The above system can be solved by finding x from the first system and replacing it to the second system to get y.

- More specifically,

$$Ax + By = a \Rightarrow A^{-1}Ax + A^{-1}By = A^{-1}a \Longrightarrow Ix + A^{-1}By = A^{-1}a \Rightarrow$$
$$x = A^{-1}a - A^{-1}By$$

Problem: Block elimination (cont.)

- The second equation becomes now:

$$C(A^{-1}a - A^{-1}By) + Dy = b \Rightarrow$$
$$CA^{-1}a - CA^{-1}By + Dy = b \Rightarrow$$
$$(D - CA^{-1}B)y = b - CA^{-1}a$$

- If we write the set of the two systems in a single matrix form we obtain:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

- The matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a **block-matrix**. This means that its elements are matrices. We refer to them as **sub-matrices**.

Problem: Block elimination (cont.)

- Our goal is to carry elimination in a block-matrix.
- We know how to do elimination in a standard matrix where the elements are scalars.
- In the case of a standard 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we can remove element a_{21} by multiplying the first row of the matrix with $-\frac{a_{21}}{a_{11}}$ and adding it to the second row. This yields the upper triangular matrix $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} \end{bmatrix}$.
- The above is equivalent to multiplying the original matrix from the left with the elimination matrix

$$\begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix}$$

Problem: Block elimination (cont.)

- Our goal is to carry elimination in a block-matrix.
- The equivalent of inversing a scalar in case of matrices is to take the inverse of a matrix.
- In the case of a 2×2 block-matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ we can remove sub-matrix $(2,1)$, which in that case is C, by multiplying the first row of the matrix from the left with $-CA^{-1}$ and adding it to the second row. This yields the upper triangular matrix $\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$.
- This is equivalent to multiplying the original block-matrix from the left with the elimination block-matrix

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

- The right-hand side of the augmented system is also multiplied with the above elimination matrix.

Problem: Block elimination (cont.)

- The entire procedure can be depicted as follows:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B & \vdots & a \\ C & D & \vdots & b \end{bmatrix} = \begin{bmatrix} A & B & \vdots & a \\ \mathbf{0} & D - CA^{-1}B & \vdots & b - CA^{-1}a \end{bmatrix}$$

- The above matrix form provides the same result that we obtained when we explicitly removed x from the second system of equations.

$$(D - CA^{-1}B)y = b - CA^{-1}a$$

- Note that the above analysis requires that the inverse of A exists.
- Can you think of the equivalent condition for the case of a single system of equations $Ax = b$?
- What happens if A is singular? In that case permutation is required.

Problem

- Solve the system of equations using block-elimination.

$$Ax + By = a$$
$$Cx + Dy = b$$
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, C = I, D = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}, a = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- **Hint:** In this problem A is not invertible and therefore, we must change the order of the two systems. That means

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A = I, B = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$D - CA^{-1}B = D - CB = \begin{bmatrix} -4 & -6 \\ -12 & -20 \end{bmatrix} \Rightarrow (D - CB)^{-1} = \begin{bmatrix} -5/2 & 3/4 \\ 3/2 & -1/2 \end{bmatrix}$$

$$y = (D - CB)^{-1}(b - CA^{-1}a) = \begin{bmatrix} -5/2 \\ 5/2 \end{bmatrix}$$

$$Ax + By = a \Rightarrow x = a - By = \begin{bmatrix} 7/2 \\ -2 \end{bmatrix}$$

Exam

- Exam questions will be of similar style and difficulty as the 6 Tutorial Sheets and also the problems in the notes.
- Make sure you have paid attention to detail and you are able to use the various algorithms you learnt in real life scenarios.
- Emphasize on:
  - ➢ The 4 subspaces.
  - ➢ The 5 decompositions: LU, QR, eigenvalue decomposition, decomposition into orthogonal matrices, SVD.
  - ➢ Projections.
  - ➢ Properties of determinants.
  - ➢ Least squares method.